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CURVATURE GROUPS OF A HYPERSURFACE

SAMUEL I. GOLDBERG[1]

ABSTRACT. A cochain complex associated with the vector 1-form determined by the first and second fundamental tensors of a hypersurface M in E^{n+1} is introduced. Its cohomology groups $H^p(M)$, called curvature groups, are isomorphic with the cohomology groups of M with coefficients in a subsheaf $\mathbb{S}_R$ of the sheaf $\mathbb{S}$ of closed vector fields on M. If M is a minimal variety, the same conclusion is valid with $\mathbb{S}_R$ replaced by a sheaf of harmonic vector fields. If the Ricci tensor is nondegenerate the $H^p(M)$ vanish. If $\mathbb{S}_R \neq \varnothing$, and there are no parallel vector fields, locally, the $H^p(M)$ are isomorphic with the corresponding de Rham groups.

1. **Introduction.** A cochain complex associated with the Levi-Civita connection of a Riemannian manifold M was studied in [1], and it was shown that its cohomology groups, called curvature groups, are isomorphic with the cohomology groups of M with coefficients in a sheaf of germs of infinitesimal homothetic transformations of M. This extended the main result of [3] concerning locally flat manifolds. Subsequent work of I. Vaisman [4] suggests that if one considers the vector 1-form σ determined by the first and second fundamental tensors of a hypersurface M in the Euclidean space E^{n+1} rather than the solder form of the bundle of frames over M, a cochain complex with similar properties may be defined. The curvature groups, that is, the cohomology groups $H^p(M)$ of this complex turn out to be isomorphic with the cohomology groups of M with coefficients in a subsheaf $\mathbb{S}_R$ of the sheaf $\mathbb{S}$ of 'closed' vector fields on M, and if M is a minimal variety then the same conclusion prevails, but $\mathbb{S}_R$ may then be replaced by a sheaf of harmonic vector fields.

Curvature is introduced by means of the integrability conditions of the Pfaff system defining the elements of $\mathbb{S}$. If the Ricci tensor is nondegenerate everywhere, then the curvature groups vanish. In particular, if M is an Einstein space with nonvanishing scalar curvature, the $H^p(M)$ are trivial. (If $n \geqslant 3$ or if $n = 2$ and M has constant sectional curvature $\neq 0$, then M is a hypersphere.) Finally, if $\mathbb{S}_R \neq 0$, and there are no parallel vector fields, locally, the $H^p(M)$ are isomorphic with the corresponding de Rham groups of $M, p = 1, \ldots, n$.

2. **The cochain complex of tensorial p-jet forms.** To avoid unnecessary duplication the notation and terminology of [1] will be employed unless otherwise indicated.

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Let M be an n-dimensional Riemannian manifold locally isometrically imbedded in E^{n+1} with the Euclidean metric. In terms of the cartesian coordinates of M let σ_{ij} denote the coefficients of the second fundamental form of M. Then the curvature of M is given by the Gauss equations

$$R_{ijkl} = \sigma_{jk}\sigma_{il} - \sigma_{jl}\sigma_{ik}, \tag{1}$$

and the Codazzi equations are

$$\nabla_k \sigma_{ij} = \nabla_j \sigma_{ik}. \tag{2}$$

If the mean curvature of the hypersurface vanishes, that is, if $g^{ij}\sigma_{ij} = 0$, where g is the metric tensor of M, then M is called a *minimal hypersurface* or a *minimal variety* of E^{n+1}. (The summation convention is employed throughout.) It is well known that a totally geodesic hypersurface is a minimal variety.

Consider the vector 1-form σ given by

$$\sigma^i = \sigma^i_j\, dx^j, \qquad \sigma^i_j = g^{ir}\sigma_{rj}.$$

From (2)

$$\nabla\sigma = 0. \tag{3}$$

Let L^p denote the $\mathfrak{F}$-module defined by the pairs (λ, α), where λ is a vector-valued p-form and α is a scalar p-form on M for $p = 1, \ldots, n$ (see [1]). For each p, an operator $D^p : L^p \to L^{p+1}$ is defined by

$$D^p(\lambda, \alpha) = (\nabla\lambda - \sigma \wedge \alpha, d\alpha).$$

If (λ, α) is D^p-closed, then α is d-exact, locally, and $\nabla^2\lambda = (\nabla(\nabla\lambda))$ is zero. Note that $(\nabla^2\lambda)^i = -\Omega^i_j \wedge \lambda^j$, where the $\Omega^i_j = R^i_{jkl}\,dx^k \wedge dx^l$ are the curvature forms of the Riemannian connection.

Let $\tilde{L}^p$ denote the submodule of L^p defined by those pairs (λ, α) with $\nabla^2\lambda = 0$. Observe that $\tilde{L}^p = L^p$ for $p = n - 1$, n; moreover, the pairs $(\sigma \wedge \varphi, \alpha) \in \tilde{L}^p$ for any scalar $(p - 1)$-form φ and p-form α on M. From (3), $D^p : \tilde{L}^p \to \tilde{L}^{p+1}$, and $D^{p+1}D^p = 0$.

In the sequel, we shall occasionally write D for D^p.

Consider the cochain complex

$$\tilde{L} = \left(\bigoplus_{p=0}^{n} \tilde{L}^p, D^p\right),$$

and assume that the Poincaré lemma holds for D, that is, on a star-shaped region (open ball) in R^n, every D-closed element of $\tilde{L}^p$, $p > 0$, is D-exact. In the sequel, we consider only complexes for which the Poincaré lemma is valid, and denote them again by $\tilde{L}$. This is certainly the case if M is locally flat. If rank $\sigma \geqslant 2$, that is, if the type number of M is $\geqslant 2$, the Poincaré lemma also holds for the subcomplex of $\tilde{L}$ obtained by considering only the pairs $(\sigma \wedge \varphi, \alpha)$.

The cohomology groups

$$H^p(\tilde{L}) = \ker D^p/\mathrm{im}\, D^{p-1}, \qquad p = 1, \ldots, n,$$

will be called the *curvature groups of the hypersurface*. We shall also write $H^p(M)$ for $H^p(\tilde{L})$, and denote ker D^0 by $H^0(M)$.

A graded ring structure is defined on $\tilde{L}$ as in [1], and D turns out to be a derivation on this ring.

3. **The sheaf of germs of c-fields.** Since the coefficients of the second fundamental form are symmetric, the system of first order partial differential equations

$$\nabla_j X^i = f\sigma_j^i, \tag{4}$$

where $X = X^i \partial/\partial x^i$ is a vector field on M and f is a smooth function, gives rise to a closed 1-form $\xi = \xi_i dx^i$, $\xi_i = g_{ik} X^k$. Hence, by the Poincaré lemma, ξ is the gradient of a function, locally. If M is a minimal variety, then X is a harmonic vector field. The vector fields characterized as solutions of (4) will be called *closed vector fields*, or simply *c-fields*. They form an additive abelian group S but not an $\mathfrak{F}$-module.

The integrability conditions of (4) yield

$$X^r R^i_{rjk} = \sigma^i_j \nabla_k f - \sigma^i_k \nabla_j f. \tag{5}$$

Contracting gives

$$X^r R_{rj} = \sigma^r_j \nabla_r f - \sigma^r_r \nabla_j f, \tag{6}$$

so if M is a minimal variety, $X^r R_{rj} = \sigma^r_j \nabla_r f$.

We are particularly interested in the case where f = constant since this yields an interpretation of the curvature groups in terms of the sheaf cohomology of the complex $\tilde{L}$. Thus, from (5), $\iota(X)\Omega = 0$, and this is satisfied if M is locally flat. On the other hand, if the Ricci tensor is nondegenerate everywhere, there are no nontrivial solutions of (4).

4. **Cohomology with coefficients in the sheaf of germs of c-fields.** Let S_R be the subspace of c-fields characterized as solutions of (4) with $f = c$ (constant). There is a monomorphism $i : S_R \to \tilde{L}^0$ defined by $i(X) = (X, c)$. Let $\mathfrak{S}_R$ denote the sheaf of germs of vector fields associated with S_R, and let $\mathfrak{L}^p, p \geqslant 0$, denote the sheaves of germs associated with the $\mathfrak{F}$-modules $\tilde{L}^p$. The mapping $D : \tilde{L}^p \to \tilde{L}^{p+1}$ induces a mapping $\mathfrak{L}^p \to \mathfrak{L}^{p+1}$ which is again denoted by D. We then have a sequence of sheaf homomorphisms

$$0 \to \mathfrak{S}_R \xrightarrow{i} \mathfrak{L}^0 \xrightarrow{D} \mathfrak{L}^1 \xrightarrow{D} \cdots \xrightarrow{D} \mathfrak{L}^n \to 0$$

which is exact. In fact, if $(X,f) \in \mathfrak{L}^0$ and $D(X,f) = (\nabla X - f\sigma, df) = 0$, then $\nabla X = c\sigma$, where f = const $= c$. Hence, $(X,f) = i(X)$. Exactness at $\mathfrak{L}^p$ for $p > 0$ is a consequence of the Poincaré lemma for D. The $\mathfrak{L}^p$ for $p = 0, 1, \ldots, n$ being fine sheaves, the sequence is a fine resolution of $\mathfrak{S}_R$. We therefore have the following interpretation of the groups $H^p(M)$.

THEOREM 1. *The curvature groups $H^p(M)$ of a hypersurface M are isomorphic with the cohomology groups $H^p(M, \mathfrak{S}_R)$ of the hypersurface with coefficients in $\mathfrak{S}_R$, $0 \leqslant p \leqslant n$.*

Corollary 1. *The curvature groups of a hypersurface with nondegenerate Ricci tensor everywhere vanish.*

Thus, if the Ricci tensor is nondegenerate everywhere then a D-closed 1-form can be expressed as $(-f\sigma, df)$.

Corollary 2. *The curvature groups of a hypersurface M with an Einstein metric of nonzero scalar curvature are trivial.*

If $n \geqslant 3$ or if $n = 2$ and M has constant sectional curvature $\neq 0$, then M is a hypersphere.

If the hypersurface is minimal, in particular, if it is totally geodesic, the curvature groups have the following interpretation.

Theorem 2. *The curvature groups of a minimal variety are isomorphic with the cohomology groups of the hypersurface with coefficients in a sheaf of germs of its harmonic vector fields.*

Corollary. *The curvature groups of a totally geodesic hypersurface are isomorphic with the cohomology groups of the hypersurface with coefficients in the sheaf of germs of its parallel vector fields.*

Note that if M is umbilical, then σ is proportional to the solder form η of the bundle of frames over M, that is, $\sigma = \mu\eta$ for some smooth function μ. The above theory is then a consequence of that developed in [1].

Theorem 3. *The curvature groups of an umbilical hypersurface M of E^{n+1}, with no flat points, are isomorphic with the curvature groups of M with coefficients in the sheaf of germs of homothetic s-fields.*

Since by the Codazzi equations, μ is actually a constant, it follows that if M is complete and connected and $\mu \neq 0$, then M is a hypersphere. If $\mu = 0$, then under the conditions, M is a hyperplane.

5. **The complex of vector forms of the type** $\sigma \wedge \varphi$. Let Φ^p be the $\mathfrak{F}$-module of vector-valued forms expressible as $\sigma \wedge \varphi$ where φ is a scalar-valued $(p-1)$-form. By (3), $\nabla(\sigma \wedge \varphi) = -\sigma \wedge d\varphi$. Thus, $\nabla(\nabla(\sigma \wedge \varphi)) = 0$. Consider the complex of forms $\Phi = (\oplus_{p=1}^n \Phi^p, \nabla^p)$ where $\nabla^p = \nabla|_{\Phi^p}$, and let $H^p(\Phi) = \ker \nabla^p / \operatorname{im} \nabla^{p-1}$ denote its pth cohomology group. Define $H^1 = \ker \nabla^1$. Assuming rank $\sigma \geqslant 2$, the chain map $\varphi \to \sigma \wedge \varphi$ establishes a bijection of the module of p-forms on M with Φ^{p+1} for $p = 0, 1, \ldots, n-1$. Clearly d-closed forms are mapped into ∇-closed forms and d-exact forms into ∇-exact forms. Thus, the p-dimensional de Rham groups of M are isomorphic with the groups $H^{p+1}(\Phi)$.

A multiplication "$\circ$" between the elements of Φ is defined by

$$\sigma \wedge \varphi \circ \sigma \wedge \psi = \sigma \wedge \varphi \wedge \psi \in \Phi^{p+q-1},$$

where φ and ψ are scalar $(p-1)$- and $(q-1)$-forms, respectively. It is an easy consequence that

$$\sigma \wedge \varphi \circ \sigma \wedge \psi = (-1)^{pq} \sigma \wedge \psi \circ \sigma \wedge \varphi,$$

$$\nabla(\sigma \wedge \varphi \circ \sigma \wedge \psi) = \nabla(\sigma \wedge \varphi) \circ \sigma \wedge \psi + (-1)^{p-1} \sigma \wedge \varphi \circ \nabla(\sigma \wedge \psi).$$

Hence, Φ is a graded ring and ∇ is a derivation on Φ. Moreover, the cohomology ring of Φ is isomorphic with the de Rham cohomology ring of M.

THEOREM 4. *Let M be a hypersurface of E^{n+1}. If the system* (4) *has a solution for some constant $c \neq 0$ but no nontrivial solution for $c = 0$, then the curvature groups $H^p(M)$ are isomorphic with the p-dimensional de Rham groups,* $1 \leqslant p < n$.

The proof is an easy consequence of Theorem 1, since by hypothesis the sheaf $\mathcal{S}_R$ is isomorphic with the sheaf of real constants.

REMARKS. (a) If M is locally convex the conditions of the theorem do not hold, as one sees from (6) and [2, p. 124].

(b) If $\nabla X = c\sigma$ has a solution for some $c \neq 0$, then it has a solution for every $c \neq 0$. We therefore have a sequence of homomorphisms

$$0 \xrightarrow{i} S_R \xrightarrow{\nabla} \Phi^1 \xrightarrow{\nabla} \Phi^2 \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} \Phi^n \xrightarrow{\nabla} 0.$$

Let $\mathfrak{F}^p$, $1 \leqslant p \leqslant n$, be the sheaves of germs associated with Φ^p. Then, the above sequence induces the sequence of sheaf homomorphisms

$$0 \xrightarrow{i} \mathcal{S}_R \xrightarrow{\nabla} \mathfrak{F}^1 \xrightarrow{\nabla} \mathfrak{F}^2 \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} \mathfrak{F}^n \xrightarrow{\nabla} 0, \tag{7}$$

so under the assumptions of Theorem 4 this sequence is exact provided $\operatorname{rank} \sigma \geqslant 2$. Since the $\mathfrak{F}^p$ are fine sheaves, the sequence (7) is a fine resolution of $\mathcal{S}_R$, and hence the groups $H^{p+1}(\Phi)$ are isomorphic with the groups $H^p(M, \mathcal{S}_R)$ for $1 \leqslant p < n$.

(c) If M is a hypersurface locally isometrically imbedded in an $(n + 1)$-dimensional space of nonzero constant curvature, the above theory may again be applied.

(d) The curvature groups can be generalized as follows: Let M be a differentiable manifold endowed with a field of endomorphisms of its tangent spaces or, equivalently, with a vector 1-form σ. If a linear connection ∇ with $\nabla\sigma = 0$ is chosen, a cochain complex $\tilde{L}$ and its 'curvature groups' can be constructed as in §2. One then seeks interesting corresponding sheaves. Many geometrical structures are then included in this scheme. (The real difficulty lies in obtaining a Poincaré lemma for D.)

PROBLEMS. (A) In the given scheme do the curvature groups depend on the connection chosen?

(B) Can one find a suitable generalization to arbitrary G-structures?

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ILLINOIS, URBANA, ILLINOIS 61801