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PRODUCTS OF SEQUENTIAL SPACES

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Dedicated to Professor Kiiti Morita
on the occasion of his 60th birthday

ABSTRACT. S. P. Franklin introduced the notion of a sequential space and characterized such spaces as being precisely the quotient images of metric spaces.

In this paper we investigate a necessary and sufficient condition for the product of a first countable space with a sequential space to be sequential, and we consider the property "sequential space" in X^{ω}.

1. **Introduction.** Throughout this paper, by a space we shall mean a regular, T_1-space.

The symbol N will refer to the set of natural numbers.

Let us recall that a space X is *sequential* [2], if a subset F of X is closed whenever $F \cap C$ is closed in C for every convergent sequence C together with its limit point.

Metric spaces, or more generally Fréchet (= Fréchet-Urysohn) spaces are sequential. Sequential spaces are k-spaces.

As is well known [2], the product of a sequential space with a separable metric space need not be sequential.

As for the product of a sequential space with a first countable space, our main theorem, which will be established in §3, reads as follows:

THEOREM 1.1. *Let X be a Fréchet space, or a sequential space each of whose points is a G_δ-set (or equivalently, a k-space each of whose points is a G_δ-set). Let Y be first countable. Then $X \times Y$ is sequential if and only if X is strongly Fréchet, or Y is locally countably compact.*

In the necessity, the property "each point of X is a G_δ-set" is essential.

According to Siwiec [12], a space X is called *strongly Fréchet* (= *countably bisequential* in the sense of E. Michael [6]) if, whenever $\{F_n; n \in N\}$ (or simply $\{F_n\}$) is a decreasing sequence accumulating at x in X, there exist $x_n \in F_n$ such that the sequence $\{x_n; n \in N\}$ (or simply $\{x_n\}$) converges to x.

Metric spaces are strongly Fréchet. Strongly Fréchet spaces are Fréchet.

As a special class of sequential spaces, we shall consider symmetrizable spaces.

According to A. V. Arhangel'skiĭ [1], a space X is *symmetrizable*, if there is a real valued, nonnegative function d defined on $X \times X$ satisfying the

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following: (1) $d(x,y) = 0$ iff (= if and only if) $x = y$, (2) $d(x,y) = d(y,x)$, and (3) $A \subset X$ is closed iff $d(x,A) > 0$ for any $x \in X - A$.

If we replace (3) by (3)′: $x \in \overline{A}$ iff $d(x,A) = 0$, then such a space X is called *semimetrizable* [4].

Metric spaces, or more generally semimetrizable spaces, are symmetrizable.

It has been shown [14] that the product of a countable, symmetrizable space with a separable metric space need not be symmetrizable.

In the following theorem, we establish a necessary and sufficient condition for the product of a symmetrizable space with a semimetrizable space to be symmetrizable.

THEOREM 1.2. *Let a symmetrizable space X be paracompact, or more generally meta-Lindelöf (i.e. every open covering has a point-countable open refinement), or have each point a G_δ-set. Let Y be semimetrizable. Then $X \times Y$ is symmetrizable if and only if X is semimetrizable, or Y is locally compact.*

As for the property "sequential space" in the product X^ω of countably many copies of X, in §4, we will have

THEOREM 1.3. *Let X have one of the three properties listed below*:

(i) *$\aleph_0$-space in the sense of E. Michael* [5],

(ii) *closed image of a metric space*,

(iii) *CW-complex in the sense of Whitehead.*

If X^ω is sequential, then X is metrizable.

It follows from [14] that there is a sequential space (in fact, a symmetrizable space) X such that X^2 is not sequential.

In this respect, it will be shown that the higher power X^ω can also behave unpredictably.

That is, *there is a space X such that X^n is sequential* (*in fact, symmetrizable*) *for all $n \in N$, but X^ω is not even sequential.*

2. **Preliminaries.** As a weaker condition than "X is strongly Fréchet", we shall often make use of the following condition (C) on X.

(C) Let $\{F_n\}$ be a decreasing sequence accumulating at $x \in X$. Then there exist $x_n \in F_n$ such that the sequence $\{x_n\}$ converges to some point $x' \in X$.

LEMMA 2.1. (A) *Let X be a Fréchet space, or a space each of whose points is a G_δ-set. If X satisfies condition* (C), *then X is strongly Fréchet.*

(B) *Let X be a symmetrizable space satisfying condition* (C). *If X is also meta-Lindelöf, or each point of X is a G_δ-set. Then X is semimetrizable.*

PROOF. (A) In case X is Fréchet, in view of the proof of [11, Theorem 5.1], X is strongly Fréchet.

In case each point of X is a G_δ-set, it may be proved directly that X is strongly Fréchet.

(B) We shall prove that X is Fréchet. Because of part (A), we need only consider the case where X is meta-Lindelöf. Let D be a countable subset of X. Then the meta-Lindelöf space $\overline{D}$ is separable, hence is Lindelöf. Since $\overline{D}$ is symmetrizable, by [10, Theorem 2] $\overline{D}$ is hereditarily Lindelöf, and hence each point of $\overline{D}$ is a G_δ-set in $\overline{D}$. Since $\overline{D}$ satisfies condition (C), by part (A), $\overline{D}$ is strongly Fréchet. Then D is Fréchet. Thus each countable subset of X is

Fréchet. Hence X is Fréchet by [6, Proposition 8.7]. Thus X is first countable, for Fréchet, symmetrizable spaces are first countable [1]. Since first countable, symmetrizable spaces are semimetrizable, X is semimetrizable.

LEMMA 2.2. *Let X be sequential. If X does not satisfy condition* (C), *then there is a countable, metric space Y_0 such that $X \times Y_0$ is not sequential.*

PROOF. Since X does not satisfy condition (C), there is a point x_0 of X, and a decreasing sequence $\{A_n\}$ accumulating at x_0 satisfying

(K) If $x_n \in A_n$, then the sequence $\{x_n\}$ has no limits. Since X is sequential and $x_0 \in \overline{A_n}$ for $n \in N$, by [6, Lemma 8.3 and Proposition 8.5], there is a sequence $\{C_n\}$ of countable subsets of X such that $C_n \subset A_n$ and $x_0 \in \overline{C_n}$.

Let $Y_0 = \bigcup_{n=1}^{\infty}\{C_n \times \{n\}\} \cup \{x_0\}$, and topologize Y_0 as follows:
Let each point of $\bigcup_{n=1}^{\infty} C_n \times \{n\}$ be open, and $\{V_n(x_0)\}$ be a countable local base at x_0, where $V_n(x_0) = \bigcup_{i \geqq n}^{\infty}\{C_i \times \{i\}\} \cup \{x_0\}$. Then Y_0 is a metric space, which is not locally compact. Let $A = \{(x,(x,n)) \in X \times Y_0; n \in N, x \in C_n\}$. Then $(x_0, x_0) \in \overline{A} - A$. Thus A is not closed in $X \times Y_0$.

Suppose that $X \times Y_0$ is sequential. Then a subset F of $X \times Y_0$ is closed whenever $F \cap (C \times K)$ is closed in $C \times K$ for every convergent sequence C in X and every convergent sequence K in Y_0. Let C, K be convergent sequences in X, Y_0 respectively, and let $B = A \cap (C \times K)$. To see that B is a closed subset of $C \times K$, let $z \in C \times K - B$. We need only consider the case $z = (x, x_0)$. The condition (K) implies that there is A_{i_0} which contains no elements of C. Then there is a neighborhood $X \times V_{i_0}(x_0)$ of z which is disjoint from the set B. Thus B is closed in $C \times K$. Hence A is closed in $X \times Y_0$, which is a contradiction. Therefore $X \times Y_0$ is not sequential.

LEMMA 2.3. *Let X be first countable. If X is not locally countably compact, then the space Y_0 in Lemma* 2.2 *is a closed subset of X.*

PROOF. By the hypotheses for X, there is a point x_0 of X, and a countable local base $\{U_n\}$ at x_0 such that each $\overline{U}_n$ is not countably compact.

By induction, we can obtain a sequence $\{C_{n_k}\}$ of countably infinite, discrete subsets of X such that $C_{n_k} \subset \overline{U}_{n_k}$, $C_{n_j} \cap C_{n_k} = \varnothing$ if $j \neq k$, and $C_{n_k} \not\ni x_0$, where $1 = n_1 < n_2 < \cdots$.

Let $Z = \bigcup_{k=1}^{\infty} C_{n_k} \cup \{x_0\}$. Then Z is a closed subset of X and is homeomorphic to the space Y_0.

K. Morita [9, Theorem 9.2] has shown that if $X \times Y$ is a Fréchet space (or equivalently, a hereditarily sequential space [3]), then X is strongly Fréchet, or Y is discrete.

As for sequential spaces, from Lemmas 2.2 and 2.3, we have

PROPOSITION 2.4. *Let X be sequential, and Y first countable. If $X \times Y$ is sequential, then X satisfies condition* (C), *or Y is locally countably compact.*

3. **Proofs of Theorems 1.1 and 1.2, and some examples.**

PROOF OF THEOREM 1.1. The necessity follows from Lemma 2.1(A) and Proposition 2.4.

The sufficiency follows from [6, Proposition 4.D.4] and [13, Corollary 2.4].

PROOF OF THEOREM 1.2. The sufficiency follows from [14, Corollary 4.4]. So we shall prove the necessity.

For this purpose, let $X \times Y$ be symmetrizable. Then $X \times Y$ is sequential, for symmetrizable spaces are sequential [1].

Suppose Y is not locally compact. Then Y is not locally countably compact, for countably compact, semimetrizable spaces are compact [10, Corollary 2]. Thus X satisfies condition (C) by Proposition 2.4. That X is semimetrizable follows from Lemma 2.1(B).

Now, by the following Remark 3.1, we see that in the necessity of the condition of Theorem 1.1, the assumptions "each point of X is a G_δ-set" and "Y is first countable" are essential.

The symbols R, Q, and Z will denote, respectively, the reals, the rationals, and the integers, all with their usual topologies.

REMARK. 3.1. (A) Let X be a compact, sequential space which is not Fréchet. In fact, such a space exists by [3, Example 7.1]. Then $X \times Q$ is sequential by [13, Corollary 2.4]. But X is not strongly Fréchet, nor is Q locally countably compact.

(B). Let X be the quotient space R/Z with Z identified to a point. Then X is a countable CW-complex. Let Y be the countable, symmetrizable space in [14, Example 3.2]. Then a subset F of Y is closed whenever $F \cap C_i$ is closed for every convergent sequence C_i $(i = 0, 1, 2, \ldots)$ in Y, where $C_0 = \{0\} \cup \{1/n; n \in N\}$, $C_i = \{1/i + 1/n; n \in N\}$. Thus, in view of the proof of [7, Lemma 2.1], $X \times Y$ is sequential. But the Fréchet space X is not strongly Fréchet, nor is Y locally countably compact.

4. The property "sequential space" in X^ω.

PROPOSITION 4.1. *Let X^ω be sequential. Then X satisfies condition* (C).

PROOF. In case X is countably compact, it is easy to check that a sequential space X satisfies condition (C).

In case X is not countably compact, the space N may be regarded as a closed subset of X. Since $X \times N^\omega$ is a closed subset of X^ω, $X \times N^\omega$ is sequential. Hence X satisfies condition (C) by Proposition 2.4.

By Lemma 2.1(A) and Proposition 4.1, we have

PROPOSITION 4.2. *Let X be Fréchet, or each point of X be a G_δ-set. If X^ω is sequential, then X is strongly Fréchet.*

LEMMA 4.3. *Let X have one of the properties* (i), (ii) *and* (iii) *in Theorem* 1.3. *If X is strongly Fréchet, then X is metrizable.*

PROOF. From [6, Theorem 9.11 and Corollary 9.10], we need only prove case (iii).

Let $\mathfrak{D} = \{e_\alpha; \alpha \in A\}$ be the collection of cells in X. We shall prove the collection $\overline{\mathfrak{D}} = \{\bar{e}_\alpha; \alpha \in A\}$ is point-finite.

Suppose that there is a point $x_0 \in X$ such that infinitely many elements of $\overline{\mathfrak{D}}$ contain the point x_0. So we may assume each $\bar{e}_{\alpha_n}$ $(n \in N)$ contains the point x_0. Let $F_n = \bigcup_{n=1}^{\infty} e_{\alpha_n} - \bigcup_{i=1}^{n} e_{\alpha_i}$. Then $x_0 \in \overline{F}_n$ for $n \in N$. Since X is strongly Fréchet, there exist $x_n \in F_n$ such that a sequence $\{x_n\}$ converges to some point $x' \in X$. Let us put $K = \{x_n; n \in N\} \cup \{x'\}$. Then K meets infinitely many elements of $\mathfrak{D}$. While, K is a compact subset of a CW-complex

X. Thus K meets only a finite number of elements of $\mathfrak{D}$. This is a contradiction. Hence the collection $\overline{\mathfrak{D}}$ is point-finite.

By a similar method, we can see that for each point x of X, assuming only these $\bar{e}_{\alpha_i}$ $(i = 1, 2, \ldots l)$ contain the point x, there is a neighborhood U of x such that $\overline{U} \subset \bar{e}_{\alpha_1} \cup \cdots \cup \bar{e}_{\alpha_l}$. Since $\bar{e}_{\alpha_1} \cup \cdots \cup \bar{e}_{\alpha_l}$ is metric, U is metric. Hence X is locally metrizable. Since a CW-complex is paracompact [9], X is metrizable.

PROOF OF THEOREM 1.3. From the hypothesis for X, each point of X is a G_δ-set. Thus, by Lemma 2.1(A), Proposition 4.1 and Lemma 4.3, X is a metric space.

From the following example, we see that the product X^ω of a sequential (symmetrizable) space X need not be sequential (symmetrizable) even if, for all $n \in N$, X^n is sequential (symmetrizable).

EXAMPLE 4.4.Let X be the symmetrizable space Y in Remark 3.1(B). Then, in view of the proof of [7, Lemma 2.1], for all $n \in N$, X^n is sequential and hence is symmetrizable by [14, Theorem 4.2], while X is an $\aleph_0$-space but is not metrizable. Then, by Theorem 1.3, X^ω is not even sequential and hence is not symmetrizable.

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