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AN ALMOST CONTINUOUS FUNCTION $f: S^n \to S^m$ WHICH COMMUTES WITH THE ANTIPODAL MAP

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ABSTRACT. It is shown that if $n, m \geqslant 1$ are integers, then there exists an almost continuous function from the n-sphere S^n onto S^m which commutes with the antipodal map.

Introduction. Hunt [1] has generalized the Borsuk-Ulam antipodal point theorem by proving that no connectivity function $f: S^n \to S^{n-1}$ commutes with the antipodal map. Since, if $n > 1$, by Corollary 1 of Stallings [5], such a function is almost continuous, it seems reasonable to ask whether Hunt's result holds for almost continuous functions. The purpose of this note is to give a counterexample.

Definitions and conventions. In the sequel we regard a function as being identical with its graph.

Suppose $f: X \to Y$. That f is *almost continuous* means that if $f \subset D$, where D is an open subset of $X \times Y$, then there exists a continuous function $g: X \to Y$ such that $g \subset D$. That K is a *minimal blocking set* of a non-almost continuous function f means that K is a closed subset of $X \times Y$, $K \cap f = \varnothing$, $K \cap g \neq \varnothing$ whenever $g: X \to Y$ is continuous, and no proper subset of K has the preceding properties.

We denote by S^n the set of all points $x = (x_1, x_2, \ldots, x_{n+1})$ of Euclidian $(n + 1)$-space R^{n+1} such that $(\sum_{i=1}^{n+1} x_i^2)^{1/2} = 1$. A function $f: S^n \to S^m$ is said to *commute with the antipodal map* if $f(-x) = -f(x)$ for each x in S^n.

The natural projection map of $X \times Y$ onto X is denoted by $p: X \times Y \to X$. The letter c denotes the cardinality of the real line.

The main results.

THEOREM 1. *Suppose $f: X \to S^m$ is not almost continuous where $m \geqslant 1$ and X is a compact metric space. There exists a minimal blocking set K of f and $p(K)$ is a perfect set.*

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PROOF. That K exists follows from Theorem 2 of [3]. Assume that z is an isolated point of $p(K)$ and let U be a neighborhood of z such that $U \cap p(K) = \{z\}$. Note that $p(K) \neq \{z\}$, because otherwise the constant map $g\colon X \to S^m$ such that $g(x) = f(z)$ would not intersect K. Thus $K - (p^{-1}(z) \cap K)$ is a closed, proper subset of K. By the minimality of K there exists a continuous function $g\colon X \to S^m$ such that $p(K \cap g) = \{z\}$. Let y be a point of S^m different from $f(z)$ and $g(z)$ and let V be a neighborhood of z such that $\overline{V} \subset U$ and $g(\overline{V}) \subset S^m - \{y\}$. Since $S^m - \{y\}$ is homeomorphic to R^m, it is an AR [4, p. 339], so the continuous function $h\colon (\overline{V} - V) \cup \{z\} \to S^m - \{y\}$, defined by $h|(\overline{V} - V) = g|(\overline{V} - V)$ and $h(z) = f(z)$, has a continuous extension $h'\colon \overline{V} \to S^m - \{y\}$. But then $g' = g|(X - V) \cup h'$ is a continuous function from X into S^m and $g' \cap K = \varnothing$, a contradiction. Thus $p(K)$ has no isolated points and is a perfect set.

THEOREM 2. *Suppose n and m are integers with $n, m \geqslant 1$. There exists an almost continuous function $f\colon S^n \to S^m$ which commutes with the antipodal map.*

PROOF. Denote by θ the set of all closed subsets T of $S^n \times S^m$ such that $\operatorname{card}(p(T)) = c$. It follows from Theorem 1 that if $f\colon S^n \to S^m$ intersects each member of θ, then f is almost continuous. Using transfinite induction in a manner quite similar to the proof of Theorem 2 of [2], for each T in θ we may choose x_T in $p(T)$ and define $f(x_T)$ and $f(-x_T)$ so that $(x_T, f(x_T))$ is in T and $f(-x_T) = -f(x_T)$. Now, if x is a point of S^n such that $f(x)$ is not defined by the above induction, neither is $f(-x)$ defined. So, for each such x, we may define $f(x)$ and $f(-x)$ arbitrarily so long as $f(-x) = -f(x)$. This completes the proof.

REFERENCES

1. J. H. V. Hunt, *A connectivity map $f\colon S^n \to S^{n-1}$ does not commute with the antipodal map*, Bol. Soc. Mat. Mexicana (2) **16** (1971), 43—45. MR **47** #9579.

2. K. R. Kellum, *Almost continuous functions on I^n*, Fund. Math. **79** (1973), 213—215. MR **47** #9598.

3. ———, *On a question of Borsuk concerning non-continuous retracts*. II, Fund. Math. (to appear).

4. K. Kuratowski, *Topology*. Vol. II, PWN, Warsaw; Acadaemic Press, New York, 1968. MR **41** #4467.

5. J. Stallings, *Fixed point theorems for connectivity maps*, Fund. Math. **47** (1959), 249—263. MR **22** #8485.

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