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AXIAL MAPS WITH FURTHER STRUCTURE

A. J. BERRICK

ABSTRACT. For $F = \mathbf{R}$, $\mathbf{C}$ or $\mathbf{H}$ an F-axial map is defined to be an axial map $\mathbf{R}P^m \times \mathbf{R}P^m \to \mathbf{R}P^{m+k}$ equivariant with respect to diagonal and trivial F^*-actions. Analogously to the real case, it is shown that $\mathbf{C}$-axial maps correspond to immersions of $\mathbf{C}P^n$ in $\mathbf{R}^{2n+k}$ while (for $F = \mathbf{R}$ and for $F = \mathbf{C}$, k odd) embeddings induce F-symmaxial maps. Examples are thereby given of symmaxial maps not induced by embeddings of $\mathbf{R}P^n$, and of $\mathbf{R}$-axial maps which are not $\mathbf{C}$-axial. Furthermore, the relationships which hold when $F = \mathbf{R}$, $\mathbf{C}$ are no longer valid for $F = \mathbf{H}$.

Let F be one of the fields $\mathbf{R}$, $\mathbf{C}$ or $\mathbf{H}$ of dimension d ($=$ 1,2,4 respectively) over $\mathbf{R}$, whose units F^* act on the right on $S(F^{n+1})$ to induce the projective space FP^n. Since the action of $\mathbf{R}^*$ extends to the action of F^*, we may regard F^* as acting also on $\mathbf{R}P^n$ and thence diagonally on $\mathbf{R}P^n \times \mathbf{R}P^n$, $n \equiv -1$ (d). By way of generalisation of the usual definitions ($F = \mathbf{R}$–see [2], [4], [12]), we say f: $\mathbf{R}P^n \times \mathbf{R}P^n \to \mathbf{R}P^{n+k}$ is *F-axial of type* (n,k) if f restricts to homotopy essential maps on the axes of the product and is equivariant with respect to the above F^*-action on its domain and trivial F^*-action on its range. If further f is homotopy equivariant—through an F^*-equivariant homotopy—with respect to interchanging the factors of the domain and trivial $\mathbf{Z}_2$-action on the range, f is *F-symmaxial*. (When $F = \mathbf{R}$ it is sometimes omitted from the notation.) This note explores the relationship between F-axial (resp. F-symmaxial) maps and the existence of an immersion (resp. embedding) of FP^n in $\mathbf{R}^m$, denoted $FP^n \subseteq (m)$ (resp. $FP^n \subset (m)$).

1. THEOREM. *Let $F = \mathbf{R}$ or $\mathbf{C}$, with $N = n$ or $(2n + 1)$ respectively.*

(a) *If $FP^n \subseteq (dn + k)$, then there exists an F-axial map of type (N, k).*

(b) *If $FP^n \subset (dn + k)$, then there exists an F-symmaxial map of type (N, k), provided k is odd if $F = \mathbf{C}$.*

(c) *If $FP^n \subset (dn + k)$, then the F-axial maps given by the constructions of* (a) *and* (b) *are homotopic through an F^*-equivariant homotopy.*

(d) *If there exists an F-axial map of type (N,k) with $2k \geqslant dn + 1$, then $FP^n \subseteq (dn + k)$.*

PROOF. (a),(d). Let γ be the realisation of the Hopf line bundle, ε the trivial real line bundle, and τ the real tangent bundle over FP^n. In the following sequence of implications, † indicates the use of the condition $2k \geqslant dn + 1$.

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$$\begin{aligned}FP^n \subseteq (dn+k) &\Leftrightarrow \tau \text{ is a subbundle of } (dn+k)\varepsilon \text{ [6]}\\ &\Leftrightarrow \tau \oplus d\varepsilon = (n+1)\gamma^* \text{ is a subbundle of }\\ &\quad (dn+k+d)\varepsilon \text{ [7, p. 100]}\\ &\dagger \Leftrightarrow \text{there exists a skew map}\\ &\quad (n+1)\gamma^* \to (d(n+1)+k)\varepsilon \text{ [5, (1.2)]}\\ &\Leftrightarrow \text{there exists a map } S^N \times S^N \to S^{N+k} \text{ which induces}\\ &\quad \text{an } F\text{-axial map of type } (N, k).\end{aligned}$$

(b) Let $f\colon F^P \to \mathbf{R}^{dn+k}$ be an embedding. (To use conventional matrix notation, we shall assume here that F^* acts on $\mathbf{R}^{dn}$ on the left.) Write $\mathbf{R}_0^m = \mathbf{R}^m \setminus \{0\}$; $\nu\colon \mathbf{R}_0^m \to S^{m-1}$, $x \mapsto x/\|x\|$; $\pi\colon S^N \to FP^N$, and set $\bar{\Delta} = \{(x,wx) \in \mathbf{R}_0^{N+1}\colon w \in F^*\}$, $\Delta' = \bar{\Delta} \cap (S^N \times S^N)$, $e = (1,0,\ldots,0) \in \mathbf{R}^{N+1+k}$, and $j\colon \mathbf{R}^{dn+k} \to \mathbf{R}^d \oplus \mathbf{R}^{dn+k}$ for the inclusion of the orthogonal complement of Fe in $\mathbf{R}^{N+1+k}$. For $u,v \in S^N$, write $a = \langle v,u\rangle_F$; and define

$$G\colon (S^N \times S^N, S^N \times S^N \setminus \Delta') \times I \to \left(\mathbf{R}^{N+1} \times \mathbf{R}^{N+1}, \mathbf{R}_0^{N+1} \times \mathbf{R}_0^{N+1} \setminus \bar{\Delta}\right),$$

$$G(u,v,t) = \begin{bmatrix} 1-|a|^2t^2 & -\bar{a}t \\ at & 1 \end{bmatrix}\begin{bmatrix} u \\ v-au \end{bmatrix};$$

$$g\colon \left(\mathbf{R}^{N+1} \times \mathbf{R}^{N+1}, \mathbf{R}_0^{N+1} \times \mathbf{R}^{N+1} \setminus \bar{\Delta}\right) \to \left(\mathbf{R}^{dn+k}, \mathbf{R}_0^{dn+k}\right),$$

$$g(x,y) = \begin{cases} \|x\|\cdot\|y\|\cdot\|f\pi\nu(x) - f\pi\nu(y)\|\cdot\left[f\pi\nu(x+y) - f\pi\nu(x-y)\right], \\ \qquad\qquad\qquad\qquad (x,y) \in \mathbf{R}_0^{N+1} \times \mathbf{R}_0^{N+1} \setminus \bar{\Delta}, \\ 0, \qquad\qquad\qquad (x,y) \in (\mathbf{R}^{N+1} \vee \mathbf{R}^{N+1}) \cup \bar{\Delta}. \end{cases}$$

Hence, define

$$F\colon S^N \times S^N \times I \to S^{N+k}, \qquad F(u,v,t) = \nu(ae + jgG(u,v,t)).$$

The reader may verify that these maps behave as required, so that $F_0\colon S^N \times S^N \to S^{N+k}$ induces an F-symmaxial map of type (N,k). (When $F = \mathbf{C}$, the involution on $\mathbf{R}P^{2n+1+k}$ given by $\pm(ae + j(z)) \mapsto \pm(\bar{a}e + j(z))$ is homotopic to the identity provided k is odd.)

(c) Clearly it suffices to establish that the tangent bundle monomorphism

$$\tau(f)\colon \tau FP^n \to \tau\mathbf{R}^{dn+k} = \mathbf{R}^{dn+k} \times \mathbf{R}^{dn+k}$$

is fibre-homotopic to

$$g'\colon \tau FP^n \to \mathbf{R}^{dn+k} \times \mathbf{R}^{dn+k}, \qquad g'([x,y].F^*) = (f\pi(x), g(x,y))$$

(f, g as in (b)), since the F-axial maps of both (a) and (b) come from composition with $G_0\colon S^N \times S^N \to \pi^*\tau FP^n$ specified in (b).

But this is evident from the following homotopy (cf. [5, Lemma 2.2]):

$$H\colon \tau FP^n \times I \to \mathbf{R}^{dn+k} \times \mathbf{R}^{dn+k},$$

$$H([x,y].F^*,t) = \left(f\pi(x), \left[f\pi\nu(x+(1-t)y) - f\pi\nu(x-(1-t)y)\right]/(1-t^2)\right).$$

(Note that, as $t \to 1$, $1 - t^2 = 2(1-t) + O((1-t)^2)$.)

By [2], the numerical condition of 1(d) is satisfied when $n > 7$ if $F = \mathbf{C}$ and may be omitted if $F = \mathbf{R}$. Thus 1(a),(d) yield that $\mathbf{C}P^n \subseteq (2n+k)$ implies

$\mathbf{R}P^{2n+1} \subseteq (2n + k + 1)$–cf. [12, (5.2)]. When $F = \mathbf{R}$, 1(b),(c) answer affirmatively a question raised in [2] (for which, I understand, Professors Feder and Gitler also have a proof); we now show the converse is not true.

2. EXAMPLE. Let n be a power of 2. Then by [8], $\mathbf{C}P^n \subset (4n - 1)$; 1(b) now implies the existence of a $\mathbf{C}$-symmaxial (and so $\mathbf{R}$-symmaxial) map of type $(2n + 1, 2n - 1)$. But [9], [10] $\mathbf{R}P^{2n+1} \not\subseteq (4n)$, so that *the existence of a symmaxial map of type* (n,k) *does not imply* $\mathbf{R}P^n \subset (n + k)$.

The next result is perhaps more predictable. Nevertheless, it illustrates the falsity of the converse to [12, (5.2)].

3. EXAMPLE. Let $n + 1 = 2^r$, where $r \equiv 2, 3\ (4)$. Then by [4] $\mathbf{R}P^{2n+1} \subseteq (4n - 2r)$; so by [11] there exists an $\mathbf{R}$-axial map of type $(2n + 1, 2n - 2r - 1)$. However, by [13], $\mathbf{C}P^n \not\subseteq (4n - 2r - 1)$, whence, from 1(c), *the existence of an* $\mathbf{R}$-*axial map of type* $(2n + 1,k)$ *does not imply the existence of a* $\mathbf{C}$-*axial map of type* $(2n + 1,k)$.

Since 1 shows that the situation for $\mathbf{R}P^n$ largely carries over to $\mathbf{C}P^n$, one might naively hope that a comparable result holds for $\mathbf{H}P^n$. However, [3,§4] casts doubt upon, and 5 below puts paid to, such hopes.

4. LEMMA. *If there exists an* $\mathbf{H}$-*axial* (*resp.* $\mathbf{H}$-*symmaxial*) *map* f *of type* $(4n + 3,k)$, *then there exists a* $\mathbf{C}$-*axial* (*resp.* $\mathbf{C}$-*symmaxial*) *map* g *of type* $(4n + 3,k)$.

PROOF. Write $\mathbf{R}^{4n+4} = \mathbf{C}^{2n+2} \oplus \mathbf{C}^{2n+2}$ which we identify with $\mathbf{H}^{n+1}$ as $\mathbf{C}^{2n+2} \oplus \mathbf{C}^{2n+2}j$. For $x_i,y_i \in \mathbf{C}^{2n+2}, i = 1,2$, f induces g by setting

$$g(\pm(x_1,x_2), \pm(y_1,y_2)) = f(\pm(x_1 + \bar{x}_2 j), \pm(y_1 + \bar{y}_2 j)),$$

since $(x_1 a + (\overline{x_2 a})j) = (x_1 + \bar{x}_2 j)a$ for $a \in \mathbf{C}^*$. If f is symmaxial then clearly g is too.

5. EXAMPLE. Let n be a power of 2. From [8], $\mathbf{H}P^n \subset (8n - 3)$. But if there were an $\mathbf{H}$-symmaxial–or even $\mathbf{H}$-axial–map of type $(4n + 3, 4n - 3)$, then by 4 above there would exist a $\mathbf{C}$-axial map of type $(4n + 3, 4n - 3)$. So by 1(c) $\mathbf{C}P^{2n+1} \subseteq (8n - 1)$, which is contradicted by [1], [13]. Hence, $\mathbf{H}P^n \subset (4n + k)$ *does not imply the existence of an* $\mathbf{H}$-*axial map of type* $(4n + 3,k)$.

As for positive results in the quaternionic case, we must content ourselves with the following observation.

6. NOTE. *If there exists an* $\mathbf{H}$-*axial map of type* $(4n + 3,k)$ *with* $2k \geqslant 4n + 1$, *then* $\mathbf{H}P^n \subseteq (4n + 3 + k)$. The proof is as for 1(c) above, save that one uses the characterisation of the tangent bundle given in [3, §4].

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St. John's College, Oxford OX13JP, England