PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 54, January 1976

A REMARK ON WHITNEY'S PROOF OF DE RHAM'S THEOREM[1]

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ABSTRACT. The PL differential forms are used to prove that the mapping induced on cohomology by pull-back of differential forms corresponds under the de Rham isomorphism to the pull-back of cohomology classes.

H. Whitney (*Geometric integration theory* [**W**]) gave a beautiful proof of de Rham's theorem. His approach, however, had a serious limitation. He did not prove that, under the de Rham isomorphism $H^*_{DR}(M) \approx H^*(M)$, the mapping induced on cohomology by the pull-back of forms corresponds to the pull-back of cohomology classes. It is the purpose of this note to fill this gap. We emphasize that the result itself is very well known and our only aim is to give a proof consistent with Whitney's proof of de Rham's theorem. The only difficulty, of course, is the fact that Whitney used simplicial cohomology, and differential forms could not be pulled back under simplicial mappings. However, if we enlarge the de Rham complex to include the so-called PL forms (see [**S**]), we can define the pull-back operation and easily prove the following

THEOREM. *Let f: $M \to N$ be a C^∞ map of compact, smooth manifolds. Then the diagram*

$$\begin{array}{ccc} H^*_{DR}(N) & \xrightarrow{f^*} & H^*_{DR}(M) \\ \downarrow\int & & \downarrow\int \\ H^*(N) & \xrightarrow{f^\sharp} & H^*(M) \end{array}$$

is commutative.

REMARK. The vertical arrows above are isomorphisms induced by integration of smooth forms over simplicial chains of C^∞ triangulations of M and N, respectively. The homomorphisms f^*, $f^\sharp$ are induced by f.

1. **De Rham complex of a simplicial complex.** Let K be a finite simplicial complex. A C^∞ q-form ω on K is a family

(1) $$\omega = \{\omega_\sigma | \sigma \text{ a closed simplex of } K, \omega_\sigma \text{ a } C^\infty\, q\text{-form on } \sigma\}$$

satisfying the following compatibility conditions: If i: $\rho \hookrightarrow \sigma$ is the inclusion

Received by the editors December 6, 1974.
AMS (MOS) subject classifications (1970). Primary 58A10.
Key words and phrases. Differential forms, de Rham isomorphism.
[1] Research supported by NSF Grant MPS-74-05913 A04.

of a face, then

$$(2) \qquad i^* \omega_\sigma = \omega_\rho .$$

The above compatibility conditions are preserved by the exterior differential d, which allows us to define $d\omega = \{d\omega_\sigma\}$, the exterior derivative of ω. Obviously, $d \circ d = 0$ and the space of all forms on K becomes a cochain complex $A^*(K)$, the de Rham complex of K. The integration

$$(3) \qquad \int_\sigma \omega = \int_\sigma \omega_\sigma,$$

where $\omega \in A^q(K)$, σ is an oriented q-simplex of K, and $0 \leqq q \leqq \dim K$ is well defined and extends to a linear map $\int$: $A^*(K) \to C^*(K)$. Stokes' theorem can be applied to show that $\int$ is a chain map. Note that, if K is the complex of a C^∞ triangulation of a smooth manifold M (see [**W**, p. 124] for definition of a C^∞ triangulation), then the de Rham complex of M, $A^*(M)$, injects into $A^*(K)$ as a subcomplex, and the integration map on $A^*(K)$ is an extension of the standard integration of smooth forms.

Now let f: $K \to N$ be a continuous map of K into a C^∞ manifold N such that for every closed simplex σ of K, $f|\sigma$ is C^∞. For such f we can define the pull-back f^*: $A^*(N) \to A^*(K)$ by

$$(4) \qquad f^*\omega = \{(f|\sigma)^*\omega\}, \qquad \omega \in A^*(N).$$

Again, if K is a C^∞ triangulation of a smooth manifold M and f is smooth, this agrees with the standard pull-back.

2. **Homotopy formula.** Let K and N be as above. Suppose F: $K \times I \to N$ ($I = [0,1]$) is a continuous mapping such that $f|\sigma \times I$ is C^∞ for every closed simplex σ of K. Let j_t be the injection of K into $K \times I$ given by $j_t(p) = (p, t)$ and let $f_t = F \circ j_t$. With this notation we have the following

LEMMA. *There exists a linear mapping H: $A^*(N) \to A^{*-1}(K)$ such that*

$$(5) \qquad f_1^*\omega - f_0^*\omega = Hd\omega - dH\omega$$

for every $\omega \in A^*(N)$.

PROOF. We observe that in the smooth category one can write an *explicit* formula for H (see [**G**, p. 178]). We apply this formula to $F|\sigma \times I$ to obtain $(H\omega)_\sigma$ for every smooth form ω on N. Namely $(H\omega)_\sigma = \int_0^1 j_t^* \circ i(\partial/\partial t) \circ (F|\sigma \times I)^*\omega$, where $i(\partial/\partial t)$ denotes the interior product with $\partial/\partial t$. Then the family $H\omega = \{(H\omega)_\sigma\}$ is a form on K and the mapping $\omega \to H\omega$ has the required properties. We leave the details of the proof to the reader.

3. **Induced mappings on cohomology.** We now prove our theorem. Let f: $M \to N$ be a smooth mapping of C^∞ manifolds. Let K and L be complexes of C^∞ triangulations of M and N, respectively. By the simplicial approximation theorem there exists a subdivision K_1 of K and a simplicial mapping φ: $K_1 \to L$ such that $f(p)$ and $\varphi(p)$ lie on a closed simplex of L for every $p \in K_1$ ($= M$). We can, therefore, define

$$(6) \qquad F(p,t) = (1-t)f(p) + t\varphi(p)$$

for $p \in M$, $t \in [0, 1]$. Note that $F|\sigma \times I$ is C^∞ for every closed simplex σ of K_1 so that the homotopy formula (4) can be applied.

Let ω be a closed q-form on N and let α be the cohomology class of the cochain defined by integrating ω. By definition, $f^\# \alpha = \varphi^\# \alpha$. The change of variable formula

$$\int_\sigma \varphi^* \omega = \int_{\varphi(\sigma)} \omega \tag{7}$$

shows that $\varphi^\# \alpha$ is represented by the cochain $\int \varphi^* \omega$. Thus, we only have to show that $\varphi^* \omega$ and $f^* \omega$ have the same periods on cycles of K_1. By the homotopy formula (5) applied to F,

$$f^* \omega = \varphi^* \omega + dH\omega, \tag{8}$$

and Stokes' theorem shows that $f^* \omega$ and $\varphi^* \omega$ indeed have equal periods.

References

[G] W. Greub, S. Halperin and R. Vanstone, *Connections, curvature, and cohomology*. Vol. I, Academic Press, New York, 1972.

[S] D. Sullivan, *Differential forms and the topology of manifolds*, Proc. Tokyo Conf. on Manifolds, 1973.

[W] H. Whitney, *Geometric integration theory*, Princeton Univ. Press, Princeton, N. J., 1957. MR **19**, 309.

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