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DECOMPOSITION OF C^∞ INTERTWINING OPERATORS FOR LIE GROUPS

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ABSTRACT. Let U be a unitary representation of a Lie group G in a Hilbert space $\mathcal{H}$ and let $C^\infty(U)$ denote the space of differentiable vectors for U given its usual topology. A continuous operator on $C^\infty(U)$ is said to be a C^∞ intertwining operator for U if it commutes with U. It is shown that if one decomposes U via a central decomposition into a direct integral of unitary representations, then every C^∞ intertwining operator decomposes into a direct integral of unique C^∞ intertwining operators. Furthermore, it is shown that if U is type I and primary, then every C^∞ intertwining operator extends to unique bounded (in the sense of $\mathcal{H}$) intertwining operator defined on all of $\mathcal{H}$.

I. **Introduction.** Let U be a unitary representation of a Lie group in a Hilbert space $\mathcal{H}$. A vector $v \in \mathcal{H}$ is said to be a C^∞ vector for U if the map $g \to U(g)v$ is a C^∞ map of G into $\mathcal{H}$. If X is an element of the Lie algebra $\mathfrak{L}$ of G, then $\partial U(X)v$ is defined to be $(d/dt)|_{t=0}(t \to U(\exp tX)v)$ for $v \in C^\infty(U)$. The mapping $X \to \partial U(X)$ extends to a representation of the enveloping algebra $\mathcal{E}$ into the operators on $C^\infty(U)$. We topologize $C^\infty(U)$ via the seminorms $\|\partial U(X)\cdot\|$, $X \in \mathcal{E}$. An operator $S : C^\infty(U) \to C^\infty(U)$ continuously is said to be a C^∞ intertwining operator if it commutes with $U|C^\infty(U)$.

Now suppose U has been decomposed into a direct integral of *primary* unitary representations U^α so that

$$U = \int_M U^\alpha \, d\mu(\alpha).$$

Then it follows from a theorem of Goodman that $C^\infty(U)$ also so decomposes as the direct integral of the $C^\infty(U^\alpha)$. It seems natural to ask if the C^∞ intertwining operators for U also so decompose. The affirmative answer to this question is the first main result of this paper.

Our ability to give an affirmative answer has an interesting consequence. Using a result of Poulsen's which says that the only intertwining operators for unitary irreducible representations are scalar multiples of the identity, we are able to show that for a type I primary representation U, every C^∞ intertwining operator is necessarily bounded in $\mathcal{H}$. Hence C^∞ intertwining operators for a type I representation are direct integrals of *bounded* operators in a canonical way.

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II. **Proofs.** Let us begin by stating more explicitly the needed results of Poulsen [4] and Goodman [1].

THEOREM A (POULSEN). *Let $\beta(\cdot, \cdot)$ be a continuous sesquilinear form on $C^\infty(U)$ which satisfies $\beta(U(x)v, U(x)w) = B(v, w)$ for all $v, w \in C^\infty(U)$. Then there exists a unique closed operator S on $\mathcal{H}$ whose domain contains $C^\infty(U)$ such that $\beta(\cdot, \cdot) = (S \cdot, \cdot)$, where $(\cdot, \cdot)$ denotes the scalar product on $\mathcal{H}$. In this case $S|C^\infty(U)$ is continuous and intertwines $U|C^\infty(U)$. If U is irreducible, $\beta = c(\cdot, \cdot)$ where $c \in \mathbf{C}$ and, hence, $S = cI$.*

One can draw several conclusions from this theorem. If S is a C^∞ intertwining operator, S possesses a formal adjoint S^* which is also a C^∞ intertwining operator. This is obtained by letting $\beta = (\cdot, S\, \cdot)$. It follows that the space of C^∞ intertwining operators is generated (as a vector space) by its formally skew-adjoint elements (or equivalently, its formally self-adjoint elements). Also, it follows from the uniqueness that if S is formally skew-adjoint on $C^\infty(U)$, then S is essentially skew-adjoint.

The result of Goodman's we shall need is

THEOREM B. *Let U be decomposed into a direct integral $\int_M U^\alpha \, d\mu(\alpha)$ where U^α are unitary representations of G realized in spaces $\mathcal{H}^\alpha$. Then $v \in C^\infty(U)$ iff $v^\alpha \in C^\infty(U^\alpha)$ for a.e. α and $\alpha \to \|\partial U^\alpha(X)v^\alpha\|^2$ is integrable for a.e. α for all $X \in \mathfrak{G}$. In this case $(\partial U(X)v)^\alpha = \partial U^\alpha(X)v^\alpha$ for a.e. α.*

We can now prove our first main result.

THEOREM 1. *Let U be decomposed as in Theorem* B *above and suppose that this is the primary decomposition. Let S be a C^∞ intertwining operator for U. Then there exist for a.e. $\alpha \in M$ unique C^∞ intertwining operators S^α for U^α such that $(Sv)^\alpha = S^\alpha v^\alpha$ for a.e. α and all $v \in C^\infty(U)$.*

PROOF. By the above comments it suffices to assume $-S^* = S$ on $C^\infty(U)$. Then S is essentially skew-adjoint and we can form the operator $V_t = \exp t(S)^-$, which by Stone's theorem is a unitary representation of $\mathbf{R}$. For all $x \in G$, $U(X)$ leaves the domain of S invariant and commutes with S on this domain. Hence $U(x)$ does the same for S^-. It follows (by forming the Caley transform of $(S)^-$, for example) that $U(x)$ commutes with the spectral projections of $(S)^-$ and, hence, that $U(x)$ commutes with V_t for all $t \in \mathbf{R}$. Thus, since we are dealing with the central decomposition of U, this implies that V decomposes into a direct integral of representations V^α of $\mathbf{R}$ in $\mathcal{H}^\alpha$ which commute with U^α.

Now, form the representations W and W^α of $\mathbf{R} \times G$ defined respectively by $W_{(t,x)} = V_t U_x$ and $W^\alpha_{(t,x)} = V^\alpha_t U^\alpha_x$. Then W is a direct integral of the W^α. From Stone's theorem, the space of C^∞ vectors for V contains the intersection of the domains of $S^n = C^\infty(U)$. Hence, from [2], $C^\infty(U) = C^\infty(W)$. On this space $(\partial/\partial t)|_{t=0} W_{(t,0)} = S$. From Goodman's Theorem B above, S is then a direct integral of the operators $S^\alpha = (\partial/\partial t)|_{t=0} W^\alpha_{(t,0)}$ on $C^\infty(W^\alpha)$. We claim that $C^\infty(W^\alpha) = C^\infty(U^\alpha)$ and that S^α is continuous on $C^\infty(U^\alpha)$ for a.e. α. It is obvious that $C^\infty(W^\alpha) \subset C^\infty(U^\alpha)$. To show the opposite inclusion let $X_i \in \mathfrak{G}(G)$ be such that $\|Sv\| \leqslant C \sum_i \|\partial U(X_i)v\|$ $(i = 1, \ldots, v)$ $\forall v \in C^\infty(U)$ (these exist due to the continuity of S). We claim that

$$\|S^\alpha v\| \leqslant C \sum_i \|\partial U^\alpha(X_i)v\| \tag{1}$$

for a.e. α and all $v \in C^\infty(W^\alpha)$.

From Theorem B it follows that if $\{v^\alpha\} \in C^\infty(W)$, then $\{f(\alpha)v^\alpha\} \in C^\infty(W)$ for all scalar functions $f \in L^\infty(M)$. From this, one sees that (1) holds for a.e. v^α where $\{v^\alpha\} \in C^\infty(W)$. However, in [3] we showed that $C^\infty(W)$ has a countable dense set v_n and that, for such a set, v_n^α is dense in $C^\infty(W^\alpha)$ for a.e. α. Hence (1) holds for a.e. α and every $v \in C^\infty(W^\alpha)$ by density. By similar reasoning $C^\infty(W^\alpha)$ is dense in $C^\infty(U^\alpha)$ for a.e. α. But this shows that S^α is continuous in $C^\infty(U^\alpha)$. Recalling that the topology of $C^\infty(W^\alpha)$ is defined by S and $\partial U(X)$, $X \in \mathcal{O}(G)$, we see that by density $C^\infty(W^\alpha) = C^\infty(U^\alpha)$. Hence S is a direct integral of the S^α on $C^\infty(U)$. The S^α are obviously unique for a.e. α.

Now we turn to our second main result.

THEOREM 2. *If U is a type* I *primary representation and S is as above, then S has a unique bounded extension to all of $\mathcal{H}$.*

PROOF. Again it suffices to assume that S is essentially skew-adjoint. Let V_t, $t \in \mathbf{R}$, and W be as before.

Let $\int_X \oplus Y^\beta \, d\eta(\beta)$ be an *irreducible* decomposition of W where Y^β are realized in $\mathcal{H}^\beta$. Since $\mathbf{R}$ is in the center of $\mathbf{R} \times G$, $Y^\beta|\mathbf{R}$ acts via multiplication by a character χ^β of $\mathbf{R}$. Hence $U^\beta = Y^\beta|G$ is irreducible and $\int_X \oplus U^\beta \, d\eta(\beta) = U$. This implies that the U^β are all equivalent to each other and are equivalent to a fixed unitary representation T realized in a Hilbert space $\mathcal{H}$ (U is type I!). It follows that U is equivalent to the representation U' realized in $L^2(X, \eta, \mathcal{H})$ defined by $U'(x)(f)(\beta) = T(x)f(\beta)$. Under this equivalence, V is mapped into multiplication by $\beta \to \chi^\beta$. Hence S is mapped onto multiplication by A^β where $A^\beta = (d/dt)|_{t=0}\chi^\beta(t)$. This will be bounded iff the essential supremum of $|A_\beta|$ is finite as β ranges over X. To see that this is so, let $v \in C^\infty(T)$ and let $f \in L^2(X, \eta)$. Then the function $u(\beta) = f(\beta)v$ is in $C^\infty(U')$ by Theorem B. Hence

$$\int_X |f(\beta)|^2 |A_\beta|^2 \|v\|^2 \, d\eta(\beta) = \|Su\|^2 < \infty.$$

This implies that multiplication by $\beta \to A_\beta$ defines a bounded transform on $L^2(X, \eta)$ and hence A_β is bounded.

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