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ANTICONFORMAL AUTOMORPHISMS OF COMPACT RIEMANN SURFACES

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ABSTRACT. We show that an automorphism of prime order of a compact Riemann surface is embeddable if it is the square of an anticonformal automorphism. Also, every embeddable automorphism of odd order of a compact Riemann surface is the square of an orientation reversing self-homeomorphism.

The results in this paper rely heavily on the work of R. Rüedy [3]. It is well known that a smooth surface embedded in $\mathbf{R}^3$ inherits a conformal structure from $\mathbf{R}^3$. We say a Riemann surface is embeddable if it is conformally equivalent to a smooth surface which is embedded in $\mathbf{R}^3$. It has recently been shown that every Riemann surface is embeddable [1], [2]. If X is a Riemann surface then an (anticonformal) automorphism of X is a (anti)conformal homeomorphism of X onto itself. If X has an automorphism f then we say f is embeddable if there is an embedding d: $X \to \mathbf{R}^3$ such that dfd^{-1} is the restriction of a rotation. Now Rüedy [3] has given necessary and sufficient conditions for an automorphism to be embeddable, and for completeness we quote his result here.

Let $\langle f \rangle$ denote the group generated by f. Let $F(f)$ denote the set of fixed points of f and $F(\langle f \rangle) = \bigcup_{j=1}^{r-1} F(f^j)$ where r is the order of f. If p is a fixed point then there exists a chart (D, ϕ) such that $\phi(p) = 0$ and $\phi f \phi^{-1}(z) = z \exp i\alpha$. Now $\alpha = \alpha(f, p)$ is unique up to a multiple of 2π, independent of the choice of chart. We normalize by requiring $-\pi < \alpha \leqslant \pi$.

THEOREM 1 (RÜEDY). *Let X be a compact Riemann surface with an automorphism f. Then f is embeddable if and only if*:

(1) $F(f) = F(\langle f \rangle)$;

(2) *the number of fixed points of f is even*;

(3) *either $\alpha(f, p) = \pi$ (i.e. $f^2 = \mathrm{id}$) or $\sum_{p \in F(f)} \alpha(f, p) = 0$*;

(4) $|\alpha(f, p)| = |\alpha(f, q)|$ *for any two fixed points p and q*.

We apply this result in the following theorem.

THEOREM 2. *Let X be a Riemann surface with an automorphism f of prime order r. Then f is embeddable if $f = g^2$, where g is anticonformal.*

PROOF. We assume that $f = g^2$, where g is anticonformal. If $r = 2$ or f has no fixed points then this is immediate, so we assume that $r > 2$ and f has fixed points. We now check each of the conditions of Theorem 1 separately.

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(1) If $f^j(q) = q$, with $1 < j < r$, then (since $(j, r) = 1$) there exist integers k and ℓ such that $kj + \ell r = 1$. Hence $f(q) = f^{kj+\ell r}(q) = f^{kj}(q) = q$. Thus $F(f) = F(\langle f\rangle)$.

(2) It is easy to show that an anticonformal automorphism whose square is not the identity has no fixed points. If p is a fixed point of f, then $g(p) \neq p$ and $g(p)$ is also a fixed point of f. Thus we may arrange the fixed points of f in pairs and the total number of them is even.

(3) Since g is anticonformal, $\alpha(f, p) = -\alpha(f, g(p))$.

(4) The proof is by induction on the number of fixed points. If there are two fixed points then this is immediate from (3).

Now assume that there are $2j$ fixed points, $j > 1$, and assume that (4) holds when there are i fixed points, where $1 \leqslant i < j$. Let p and $g(p)$ be a pair of fixed points of f. Then there exists a pair of open disks D and D' in X which have centers at p and $g(p)$, respectively, such that f maps each disk onto itself and such that $g(D) = D'$. Notice that $\alpha(p, f)$ is completely determined by the action of g on ∂D. Now glue the two boundary curves together by identifying $x \in \partial D$ with $g(x) \in \partial D'$. This is again a Riemann surface and f acts on this surface as a holomorphic map. By the induction hypothesis this surface satisfies the conditions of Theorem 1 so that X can be embedded in $\mathbf{R}^3$ in such a way that f is the restriction of a rotation. Now let γ be the curve obtained by identifying ∂D and $\partial D'$. Clearly f maps γ onto itself and the angle of rotation of γ is the same as $\pm\alpha(q, f)$, where $q \neq p, f(p)$ and q is any one of the remaining fixed points of f. Thus $|\alpha(p, f)| = |\alpha(q, f)|$, for all $q \in F(f)$.

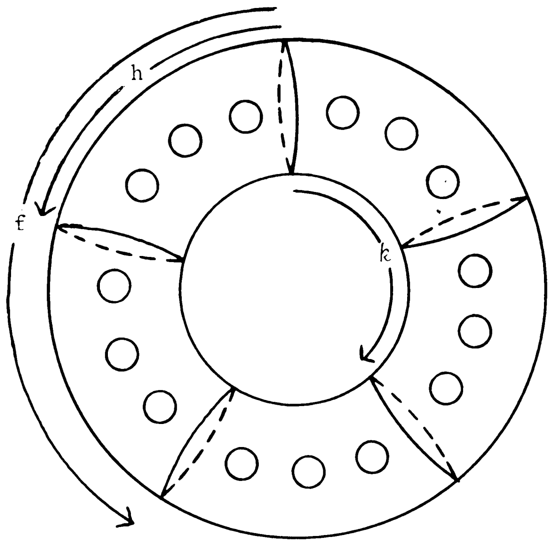
As a partial converse we have the following.

THEOREM 3. *If X is Riemann surface with an embeddable automorphism f of odd order r, then f is the square of an orientation reversing self-homeomorphism.*

PROOF. It suffices to consider the situation in which X and f are both smooth. We embed X in $\mathbf{R}^3$ so that f is the restriction of a rotation through an angle α. If f is fixed point free the situation is as in Figure 1. By [3, Lemma 1], there exists a simple closed curve γ such that $X - \bigcup_{i=1}^r f^i(\gamma)$ consists of r components which are permuted by f. The closure of each component is a sphere with two boundary components and n handles. Let $\beta = 2\pi/r$ and let h be a rotation through an angle of β so that h is a self-homeomorphism of X. Thus $h^m = f$ for some m, where $1 \leqslant m < r$. We may alter the embedding of X in $\mathbf{R}^3$, if necessary, so that (1) X is invariant under a rotation k of angle $\beta/2$ or $\beta(r + 1)/2$, depending on whether n is even or odd, and (2) X is invariant under a vertical reflection j which commutes with k. Then $k^2 = h$ and $f = (k^m j)^2$.

In the case in which there are fixed points an analogous proof holds. Here one uses [3, Lemma 2].

REMARK. If, in Theorem 3, we allow f to have even order then only partial results are known. If f has even order greater than 2, then f is the square of an orientation reversing map provided n is even. (We define n in the proof of Theorem 3.) If X has genus ℓ and f is of order 2 with 0 (resp. $2k + 2$, $\ell > k \geqslant 0$) fixed points then f is the square of an orientation reversing map if $\ell \equiv 1 \bmod 4$ (resp. $\ell \equiv 3k \bmod 4$). These facts may be shown by methods



X

(with r = 5, n = 3, m = 2)

FIGURE 1

similar to those used in proving Theorem 3. Finally, it is shown in [4] that if f is the hyperelliptic involution then it again satisfies the conclusion of Theorem 3 iff the genus of X is even. I do not know if these are the only cases in which an automorphism of order 2 is the square of an orientation reversing map.

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