

SOME METRIZATION THEOREMS

H. H. HUNG¹

ABSTRACT. We prove, using H. W. Martin's result on metrizable symmetric spaces and a symmetric of P. W. Harley III's construction, a theorem which is slightly stronger than a recent theorem of Nagata.

In this paper we give some metrization theorems which are all stronger than the now classical theorem of Nagata, Smirnov and Bing [8], [11] and [2]. The main theorem from which all the others are deduced is in fact a *little* stronger than a theorem² of Nagata's implicit in his second proof of the classical theorem [10], as pointed out only very recently by H. W. Martin [7], which has the following formulation.

THEOREM (NAGATA). *A topological space X is metrizable if and only if it is T_3 and has a base \mathcal{G} with the following properties:*

- (i) $\mathcal{G} = \bigcup_{i \in \mathbf{N}} \mathcal{G}_i$;
- (ii) for each $i \in \mathbf{N}$, \mathcal{G}_i is conservative;
- (iii) for each $i \in \mathbf{N}$ and each $x \in X$, the set $\bigcap \{A : x \in A \in \mathcal{G}_i\}$ is a neighbourhood of x .

That² Nagata's Theorem above is used in a very recent paper of Martin's [7] to prove a recent Metrization Theorem of Burke, Engelking and Lutzer [3] indicates to some extent the importance and the power of the theorems of Nagata's and ours.

To prove our main theorem, we make use of a recent result of Martin on metrizable symmetric spaces [6] which is an improvement of an earlier one of Arhangel'skiĭ [1], in the way Harley [4] used it to prove the classical Nagata-Smirnov Metrization Theorem, only slightly more efficiently perhaps, and come to the following conclusion, among others.

THEOREM . *A topological space X is metrizable if and only if it is T_3 and has a base \mathcal{Q} with the following properties:*

- (i) $\mathcal{Q} = \bigcup_{i \in \mathbf{N}} \mathcal{Q}_i$;
- (ii) for each $i \in \mathbf{N}$ and each $x \in X$, the set

$$\bigcap \{ \text{Int Cl } A : x \in A \in \mathcal{Q}_i \} \cap \bigcap \{ \infty \text{Cl } A : x \in \infty \text{Cl } A, A \in \mathcal{Q}_i \}$$

is a neighbourhood of x .

Received by the editors September 26, 1974.

AMS (MOS) subject classifications (1970). Primary 54E35, 54D25.

Key words and phrases. Symmetric spaces, metrization.

¹ This research had been supported by (Canadian) NRC grant A8748, until May, 1974 when the author left Canada and went into the service of His Malaysian Majesty, the Agung.

² These facts were provided by the referee when he or she reported on this paper.

Quite clearly, a σ -locally finite base is such a base. In fact, it suffices if each \mathcal{Q}_i is point finite and conservative. Also quite clearly, the base of Nagata's is such a base.

The two theorems, Nagata's and ours, are so nearly the same that one may suspect them to be equivalent, in the sense that our base can be, with a small amount of work, tinkered into one of Nagata's description. But there is a fundamental difference. While it is true that the requirement that for each $i \in \mathbf{N}$ and each $x \in X$, the set

$$\bigcap \{ \infty \text{Cl } A : x \in \infty \text{Cl } A, A \in \mathcal{Q}_i \}$$

is a neighbourhood of x is equivalent to the requirement that for each $i \in \mathbf{N}$, the family \mathcal{Q}_i is conservative; clearly, in general, that for each x the set

$$\bigcap \{ \text{Int Cl } A : x \in A \in \mathcal{Q}_i \}$$

is a neighbourhood of x does not *necessarily* mean that for each x the set

$$\bigcap \{ A : x \in A \in \mathcal{Q}_i \}$$

is also a neighbourhood of x . Nor, in general, does it mean that for each x the set

$$\bigcap \{ \text{Int Cl } A : x \in \text{Int Cl } A, A \in \mathcal{Q}_i \}$$

is also a neighbourhood of x .

For example, let

$$\begin{aligned} A_n &= (0, 1) \cup (1, 1 + 1/n) \quad \text{for all } n \in \mathbf{N}, \\ \mathcal{Q}_i &= \{ A_n : n \in \mathbf{N} \}, \end{aligned}$$

on \mathbf{R} .

Clearly it is true that for each x , $\bigcap \{ \text{Int Cl } A : x \in A \in \mathcal{Q}_i \}$ is a neighbourhood of x , while it is *not* true that $\bigcap \{ \text{Int Cl } A : 1 \in \text{Int Cl } A, A \in \mathcal{Q}_i \}$ is a neighbourhood of 1.

In §1 below, we construct again Harley's symmetric on a topological space and prove the sufficiency as Harley did of the condition for metrizability of Nagata and Smirnov's. In our proof, the compatibility of the symmetric topology is more on the surface as it were and possible weakening of the hypothesis actually beckons the beholder. In §2, we take up the invitation to weaken that hypothesis and arrive at a new theorem, Theorem 2.1, stronger than the classical Nagata-Smirnov Theorem that was obtained by Harley. In §3, we weaken Theorem 2.1 in various ways and give a few more theorems.

1. Preliminaries. For the definitions of a symmetric and a symmetric space, the reader is referred to [1], [4], [6]. Briefly, a symmetric is that which if it also satisfies the usual triangle inequality is also a metric. A symmetric space is a space the topology of which consists of those (and only those) sets that contain a ball of some radius around every one of their members. Such a topology is said to be induced by the symmetric onto the space.

On any topological space, we say a subset B separates $x, y \in X$ if either $x \in B, y \in \infty\text{Cl } B$ or $y \in B, x \in \infty\text{Cl } B$; a family \mathfrak{B} of subsets separates $x, y \in X$ if there is at least one member of \mathfrak{B} that separates them.

Given any Hausdorff space X and any (open) base \mathcal{A} which is $\bigcup_{i \in \mathbb{N}} \mathcal{A}_i$, we can define a nonnegative real valued function ρ on $X \times X$ as follows. For all $x, y \in X, x \neq y$, we can define $\rho(x, y)$ such that $1/\rho(x, y) =$ smallest i for which \mathcal{A}_i separates x, y ; which is always possible as long as X is Hausdorff and \mathcal{A} is a base. For all $x \in X, \rho(x, x)$ is defined to be 0. Such a ρ is obviously a symmetric and we refer to it in the following as the symmetric of Harley.

If X is T_3 and \mathcal{A} is a σ -locally finite (open) base, as in the hypothesis of the Nagata-Smirnov-Bing Metrization Theorem, then the symmetric of Harley on X can be proved to induce precisely the topology that has always been on X . The symmetric of Harley constructed out of the base $\mathcal{A} = \bigcup_{i \in \mathbb{N}} \mathcal{A}_i$ and that constructed when \mathcal{A} is considered equal to $\bigcup_{i \in \mathbb{N}} \{ \bigcup_{j \leq i; j \in \mathbb{N}} \mathcal{A}_j \}$ being identical; we can, with no loss of generality, in our proof of the compatibility of the induced topology, assume that $\mathcal{A}_i \subset \mathcal{A}_{i+1}$, for $i \in \mathbb{N}$.

To prove that the symmetric of Harley induces a sufficiently large topology, it suffices to produce, for every $y \in X$ and every open neighbourhood A of y , a ball of some finite radius r centered at $y, N(y, r) \equiv \{x \in X: \rho(x, y) < r\}$, totally within A . To prove that the topology so induced is not excessively large and, therefore, just right, we need only exhibit, for any ball of any (finite) radius about any point, an open neighbourhood of the point within that ball. For our first task, we note that the regularity of X guarantees the existence of such a $B \in \mathcal{A}$ that $x \in B \subset \text{Cl } B \subset A$ and if this B is a member of \mathcal{A}_i , then the ball $N(x, 1/i)$ is clearly within $\text{Cl } B$ and, therefore, A . For our second task, we note that, for all $x \in X, i \in \mathbb{N}$, the set

$$E_{x,i} \equiv \bigcap \{ \text{Cl } A : x \in A \in \mathcal{A}_i \} \setminus \bigcup \{ A : x \notin \text{Cl } A, A \in \mathcal{A}_i \}$$

contains x and is within $N(x, 1/i)$. That $E_{x,i}$ is a neighbourhood of x is because it clearly contains the set

$$F_{x,i} \equiv \bigcap \{ \text{Int } \text{Cl } A : x \in A \in \mathcal{A}_i \} \setminus \bigcup \{ \text{Cl } A : x \notin \text{Cl } A, A \in \mathcal{A}_i \},$$

which is open as the intersection is finite and the union over a conservative family. The space X can therefore be considered a symmetric space.

On this X , any compact set disjoint from a closed set can be covered by a finite number of basic open sets the closures of all of which are disjoint from that same closed set. These covering basic open sets, finite in number, all figure prominently among members of the family $\bigcup_{i < n; i \in \mathbb{N}} \mathcal{A}_i$ (for some large enough n), which therefore certainly separates points x of the compact set from points y of the closed set—indeed $\rho(x, y) > 1/n$ according to Harley. Martin's Theorem therefore applies to X and metrizability follows.

2. Main result. In the preceding section, clearly, the validity of Harley's symmetric follows *only* from X being Hausdorff. This symmetric always induces a sufficiently large topology (to coincide with that on X) as long as X is regular and \mathcal{A} is a base. The induced topology is just right so long as $E_{x,i}$ is a neighbourhood for every $x \in X$ and every $i \in \mathbb{N}$. Martin's Theorem always

applies to a symmetric space *when* the symmetric is Harley's and the space is regular. We therefore have the following theorem.

2.1. THEOREM. *A topological space X is metrizable if and only if it is T_3 and has a base \mathcal{Q} with the following properties:*

- (i) $\mathcal{Q} = \bigcup_{i \in \mathbf{N}} \mathcal{Q}_i$;
- (ii) for each $i \in \mathbf{N}$ and each $x \in X$,

$$\bigcap \{ \text{Int Cl } A : x \in A \in \mathcal{Q}_i \} \cap \bigcap \{ \infty \text{Cl } A : x \in \infty \text{Cl } A, A \in \mathcal{Q}_i \}$$

is a neighbourhood of x .

3. The conditions in Theorem 2.1 can be strengthened in the interest of simplicity in formulation as follows.

3.1. THEOREM. *A topological space is metrizable if and only if it is T_3 and has a base \mathcal{Q} which is $\bigcup_{i \in \mathbf{N}} \mathcal{Q}_i$ with any one of the following properties.*

For each $i \in \mathbf{N}$,

- (i) the family $\tilde{\mathcal{Q}}_i \equiv \{A, \infty \text{Cl } A : A \in \mathcal{Q}_i\}$ is conservative;
- (ii) \mathcal{Q}_i is conservative and for each x , the set $\bigcap \{A : x \in A \in \mathcal{Q}_i\}$ is a neighbourhood of x (Nagata);
- (iii) arbitrary intersections of members of $\tilde{\mathcal{Q}}_i$ are open;
- (iv) \mathcal{Q}_i is conservative and arbitrary intersections of A_i are open;
- (v) \mathcal{Q}_i is point finite and conservative;
- (vi) \mathcal{Q}_i is locally finite (Nagata-Smirnov);
- (vii) \mathcal{Q}_i is pairwise disjoint and conservative;
- (viii) \mathcal{Q}_i is discrete (Bing).

Theorem 3.1(iii) readily gives an embedding theorem, similar to Kowalsky's embedding into a countable product of hedgehogs [5]. It also gives a neighbourhood characterization, an example of which is Nagata's [9].

We give the embedding theorem as follows without proof.

3.2. THEOREM. *A topological space is metrizable if and only if it is T_3 and is homeomorphic to a subspace of the space³ $\prod_{i \in \mathbf{N}} [T^{\alpha_i}]$.*

It is interesting to note that in the preceding theorem none of the factors of the product need, in general, even be T_1 or regular. In fact, if the metrizable space is connected, the projections of its homeomorphic image are never T_1 or regular.

REFERENCES

1. A. V. Arhangel'skii, *Mappings and spaces*, Uspehi Mat. Nauk **21** (1966), no. 4 (130), 133—184 = Russian Math. Surveys **21** (1966), no. 4, 115—162. MR **37** #3534.
2. R. H. Bing, *Metrization of topological spaces*, Canad. J. Math. **3** (1951), 175—186. MR **13**, 264.
3. D. Burke, R. Engelking and D. Lutzer, *Hereditarily closure preserving collections and metrization* (to appear).

³ The space T is that of 3 points n, o and p whose topology consists of the sets $\{n\}$, $\{p\}$, $\{n, o, p\}$ and \emptyset . For all infinite cardinals α , $[T^\alpha]$ is the Cartesian product of α copies of T with the box topology.

4. P. W. Harley III, *Metric and symmetric spaces*, Proc. Amer. Math. Soc. **43** (1974), 428—430. MR **49** #1486.
5. H. J. Kowalsky, *Einbettung metrischer Räume*, Arch. Math. **8** (1957), 336—339. MR **19**, 971.
6. H. W. Martin, *Metrization of symmetric spaces and regular maps*, Proc. Amer. Math. Soc. **35** (1972), 269—274. MR **46** #2648.
7. ———, *A note on the Frink metrization theorem*, Rocky Mountain J. Math. (to appear).
8. J. Nagata, *On a necessary and sufficient condition for metrizability*, J. Inst. Polytech. Osaka City Univ. Ser. A Math. **1** (1950), 93—100. MR **13**, 264.
9. ———, *A contribution to the theory of metrization*, J. Inst. Polytech. Osaka City Univ. Ser. A **8** (1957), 185—192. MR **20** #4256.
10. ———, *Modern general topology*, Bibliotheca Math., vol. 7, North-Holland, Amsterdam; Wolters-Noordhoff, Groningen; Interscience, New York, 1968. MR **41** #9171.
11. Ju. M. Smirnov, *A necessary and sufficient condition for metrizability of a topological space*, Dokl. Akad. SSSR **77** (1951), 197—200. (Russian) MR **12**, 845.

SCHOOL OF MATHEMATICAL SCIENCES, UNIVERSITI SAINS MALAYSIA, PULAU PINANG, MALAYSIA