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# $(CA)$ TOPOLOGICAL GROUPS

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ABSTRACT. A locally compact topological group $G$ is called $(CA)$ if the group of inner automorphisms of $G$ is closed in the group of all bicontinuous automorphisms of $G$. We show that each non-$(CA)$ locally compact connected group $G$ can be written as a semidirect product of a $(CA)$ locally compact connected group by a vector group. This decomposition yields a natural dense imbedding of $G$ into a $(CA)$ locally compact connected group $P$, such that each bicontinuous automorphism of $G$ can be extended to a bicontinuous automorphism of $P$. This imbedding and extension property enables us to derive a sufficient condition for the normal part of a semidirect product decomposition of a $(CA)$ locally compact connected group to be $(CA)$.

1. **Introduction.** The purpose of this paper is to extend the results in Zerling [5] to the case of locally compact connected groups.

If $G$ and $H$ are topological groups and $\varphi$ is a one-to-one continuous homomorphism from $G$ into $H$, $\varphi$ will be called an imbedding. $\varphi$ will be called closed or dense as $\varphi(G)$ is closed or dense in $H$. For any topological group $G$, we let $Z(G)$ and $G_0$ denote the center of $G$ and the identity component group of $G$, respectively.

If $G$ is a locally compact group, $A(G)$ will denote the topological group of all bicontinuous automorphisms of $G$, topologized with the generalized compact-open topology. $G$ will be called $(CA)$ if $I(G)$, the subgroup of $A(G)$ consisting of all inner automorphisms of $G$, is closed in $A(G)$.

If $G$ and $H$ are locally compact connected groups and $\varphi\colon G \to H$ is a dense imbedding, then $\varphi(G)$ is normal in $H$, and the mapping $\rho_G\colon H \to A(G)$ defined by $\rho_G(h)(g) = \varphi^{-1}(h\varphi(g)h^{-1})$ is a continuous homomorphism [2]. If $\varphi$ is a closed imbedding, such that $\varphi(G)$ is normal in $H$, then $\rho_G$ is clearly continuous.

For any locally compact group $H$ we let $I_H(h)$ denote the inner automorphism of $H$ determined by $h \in H$. More generally, if $A$ is a subset of $H$, $I_H(A)$ will denote the set of all inner automorphisms of $H$ determined by the elements of $A$. $I_H(H)$ will be written as $I(H)$, and the continuous homomorphism $h \mapsto I_H(h)$ of $H$ onto $I(H)$ will be denoted by $I_H$.

If $N$ is a locally compact connected group and $\psi\colon H \to A(N)$ is a continuous homomorphism of some connected topological group $H$ into $A(N)$, then $N \circledS H$ will denote the semidirect product of $N$ by $H$, which is determined by $\psi$. On the other hand, if $G$ is a locally compact connected group containing a closed normal connected subgroup $N$ and a closed connected subgroup $H$, such that $G = NH$, $N \cap H = \{e\}$, and such that the restriction of $\rho_N$ to $H$ is one-to-one, then whenever we write $\rho_N(H)$ it will be understood that the

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topology is the unique locally compact topology for which $\rho_N\colon H \to \rho_N(H)$ is bicontinuous. Therefore, we will frequently write $N \circledS H$ or $N \circledS \rho_N(H)$ for $G$.

Our main results are stated in Theorems 2.1–2.3.

2. **Main results.** Suppose that $G$ is a locally compact connected group. Then we can find a neighborhood $U$ of the identity such that $U = K \times L_1^*$, where $L_1^*$ is a local Lie group and $K$ is a compact group. Moreover, $K$ is normal in $G$, and $G = KL^*$, $[K, L^*] = \{e\}$, where $L^*$ is the group generated by $L_1^*$ in $G$ (cf. Yamabe [3], [4]).

Let $L$ denote the uniquely determined connected Lie group such that $i\colon L \to L^*$ is a continuous isomorphism of $L$ onto $L^*$. Let $D^* = L^* \cap K$, and let $D = i^{-1}(D^*)$ in $L$. Since $K \cap L_1^* = \{e\}$, $D$ is a discrete normal subgroup of $L$, and, therefore, a central subgroup of $L$.

The mapping $K \times L \to G$ defined by $(k, 1) \mapsto k \cdot i(1)$ is an open continuous homomorphism of $K \times L$ onto $G$. Therefore, $G \cong (K \times L)/\tilde{D}$, where $\tilde{D} = \{(i(d^{-1}), d)\colon d \in D\}$. $G = KL^*$ will be called a canonical decomposition of $G$.

Lemma 2.1. *Let $G$ be a locally compact connected group and let $G = KL^*$ be a canonical decomposition of $G$. Then $G$ is $(CA)$ if and only if $L$ is $(CA)$.*

Proof. We have $G \cong (K \times L)/\tilde{D}$. Let $A_D(L)$ denote the identity component group of the group of bicontinuous automorphisms of $L$ which leave $D$ elementwise fixed. From Goto [1] we see that $A_D(L)$ is a closed connected Lie subgroup of $A(L)$ and $A_0(G) = I_G(K_0) \times B$, where $B = \{\sigma\colon \sigma \in A_0(G), \sigma(x) = x, x \in K_0\}$. Let $f\colon A_D(L) \to A(G)$ be defined by $f(\sigma)(k, 1)\tilde{D} = (k, \sigma(1))\tilde{D}$. Again from Goto [1] we see that $f$ is a bicontinuous isomorphism into $B$. Hence, $f(A_D(L))$ is closed in $A(G)$, since it is locally compact in the relative topology. Since

$$A(G) \supset I_G(K_0) \times f(A_D(L)) \supset I_G(K_0) \times f(I(L)) = I(G),$$

we see that $G$ is $(CA)$ if and only if $L$ is $(CA)$.

Theorem 2.1. *Let $G$ be a non-$(CA)$ locally compact connected group. Then we can find a $(CA)$ locally compact connected group $N$, a toral group $T$, which can be imbedded in $A(N)$, and a dense vector subgroup $V$ of $T$, such that:*

(i) *$P = N \circledS T$ is $(CA)$.*

(ii) *$G \cong N \circledS V$.*

(iii) *$Z(G)$ is contained in $N$.*

(iv) *$Z_0(G) = Z_0(P)$, and $\pi(Z(P))$ is finite where $\pi$ is the natural projection of $P$ onto $T$. Moreover, if $G/Z(G)$ is homeomorphic to Euclidean space, then $Z(G) = Z(P)$.*

Proof. Let $G = KL^*$ be a canonical decomposition of $G$. Since $G$ is non-$(CA)$, $L$ will be non-$(CA)$ by Lemma 2.1. Hence there exists a $(CA)$ analytic group $H$, which contains $L$ as a dense analytic subgroup, such that the following properties stated in the main structure theorem of Zerling [5] are satisfied:

(1) $H = M \circledS T$, where $M$ is a $(CA)$ analytic group and $T$ is a toral group in $A(M)$.
(2) $L \cong M \circledS V$, where $V$ is a dense vector subgroup of $T$.
(3) $Z(L)$ is contained in $M$.
(4) $Z_0(L) = Z_0(H)$, and $\pi'(Z(H))$ is finite, where $\pi'$ is the natural projection of $H$ onto $T$. Moreover, if $L/Z(L)$ is homeomorphic to Euclidean space, then $Z(L) = Z(H)$.
(5) Each bicontinuous automorphism $\sigma$ of $L$ can be extended to a bicontinuous automorphism $\epsilon'(\sigma)$ of $H$, such that $\epsilon'\colon A(L) \to A(H)$ is a closed imbedding.

Let $j\colon K \times L \to K \times H$ be the dense imbedding induced by the dense imbedding of $L$ into $H$. Then since $D$ is central in $L$, and since the center of $L$ is contained in $M$ by $(*)$, we see that $j(\tilde{D})$ will be a discrete central subgroup of $K \times H$. We have $G \cong (K \times L)/\tilde{D}$ and we let $(K \times L)/\tilde{D} \to (K \times H)/\tilde{D}$ be the dense imbedding induced by $j$, where we have identified $\tilde{D}$ and $j(\tilde{D})$. Let

$$P = (K \times H)/\tilde{D} = (K \times (M \circledS T))/\tilde{D}.$$

Because $\{k\tilde{D}\colon k \in K\} \cdot \{h\tilde{D}\colon h \in H\}$ is a canonical decomposition of $P$, with $H \to \{h\tilde{D}\colon h \in H\}$ being a continuous isomorphism onto, we see from Lemma 2.1 that $P$ is a $(CA)$ locally compact connected group.

Since $D$ is contained in $M$, we see that $\tilde{D}$ is contained in $K \times M$. Consider $\omega\colon T \to A((K \times M)/\tilde{D})$, where $\omega(\tau)(k,m)\tilde{D} = (k,\tau(m))\tilde{D}$. $\omega$ is a well-defined imbedding, since each $\tau \in T$ keeps the center of $L$ elementwise fixed. Now consider $((K \times M)/\tilde{D}) \circledS T$ and let

$$\Delta\colon (K \times (M \circledS T))/\tilde{D} \to ((K \times M)/\tilde{D}) \circledS T,$$

where $\Delta((k,(m,\tau))\tilde{D}) = ((k,m)\tilde{D},\tau)$. $\Delta$ is a well-defined bicontinuous isomorphism onto $((K \times M)/\tilde{D}) \circledS T$. Hence,

$$P \cong ((K \times M)/\tilde{D}) \circledS T \quad \text{and} \quad G \cong ((K \times M)/\tilde{D}) \circledS V.$$

We let $N = (K \times M)/\tilde{D}$. Since $M$ is $(CA)$ from $(*)$, we have that $N$ is $(CA)$ from Lemma 2.1.

Since $M$ is connected and $D$ is discrete, it is easy to see that

$$Z(G) = (Z(K) \times Z(L))/\tilde{D} \cong \{((k,m)\tilde{D},e)\colon k \in Z(K), m \in Z(L)\},$$

which is contained in $N$. In the same way we see that

$$Z(P) = \{((k,m)\tilde{D},\tau)\colon k \in Z(K), (m,\tau) \in Z(H)\}.$$

Now let $\pi$ denote the natural projection of $P$ onto $T$. Since there are only finitely many $\tau \in T$ such that $(m,\tau) \in Z(H)$ from $(*)$, it is clear that $\pi(Z(P))$ is finite.

If $((k,m)\tilde{D},\tau)$ is in $Z_0(P)$, then, since $\pi(Z_0(P))$ is both connected and finite, $Z_0(P)$ is contained in $N$. Hence, $Z_0(G) = Z_0(P)$ because $G$ is dense in $P$.

Suppose that $G/Z(G)$ is homeomorphic to Euclidean space. Since

$$G/Z(G) \cong ((K \times L)/\tilde{D})/((Z(K) \times Z(L))/\tilde{D}) \cong (K/Z(K)) \times (L/Z(L)),$$

we see that $L/Z(L)$ is homeomorphic to Euclidean space. Therefore, $Z(L) = Z(H)$ from (*). Hence, $Z(G) = Z(P)$. This completes the proof of our theorem.

LEMMA 2.2. *Maintaining the notation in the statement and proof of Theorem 2.1, we have that* $\rho_G\colon T \to A(G)$ *is one-to-one and* $\rho_G(P) = \overline{(I(G))}$.

PROOF. We first show that $I_G(N)$ is closed in $A(G)$. To this end let $A_D(L)$ denote the identity component group of the subgroup of $A(L)$ composed of elements which leave every element of $D$ fixed. $A_D(L)$ is a closed analytic subgroup of $A(L)$ by Goto [1]. Clearly $I_L(M)$ is contained in $A_D(L)$. Moreover, we may appeal to the proof of Theorem 2.1 in Zerling [5] to see that $I_L(M)$ is actually closed in $A_D(L)$.

Let $f\colon A_D(L) \to A(G)$ be given by $f(\sigma)((k,1)\tilde{D}) = (k,\sigma(1))\tilde{D}$.

Again from Goto [1] we have that $f$ is a bicontinuous isomorphism into $A(G)$. Therefore, $f(A_D(L))$ is locally compact and so it is closed in $A(G)$.

Since direct computation shows that $I_G((k,m)\tilde{D}) = I_G(k) \cdot f(I_L(m))$, $k \in K$, $m \in M$, we have $I_G(K) \cdot f(I_L(M)) = I_G(N)$. Since $I_L(M)$ is closed in $A_D(L)$, and $f$ is a bicontinuous isomorphism into $A(G)$, and $f(A_D(L))$ is closed in $A(G)$, we see that $I_G(N)$ is closed in $A(G)$.

$\rho_G$ is one-to-one on $T$ because $\tau(n,v)\tau^{-1} = (\tau(n),v)$ for $n \in N$, $v \in V$, $\tau \in T$.

Now $\rho_G(P) = \rho_G(N) \cdot \rho_G(T) = I_G(N) \cdot \overline{I_G(V)}$. Since $I_G(N)$ is closed in $A(G)$ and $\overline{I_G(V)}$ is compact, we see that $\overline{I(G)} = I_G(N) \cdot \overline{I_G(V)} = \rho_G(P)$.

THEOREM 2.2. *Maintaining the notation in the statement and proof of Theorem 2.1, we have that each bicontinuous automorphism* $\sigma$ *of* $G$ *can be extended to a bicontinuous automorphism* $\epsilon(\sigma)$ *of* $P$. *Moreover, if* $Z(K)$ *is totally disconnected, then* $\epsilon\colon A_0(G) \to A_0(P)$ *is a bicontinuous isomorphism onto a closed subgroup of* $A(P)$.

PROOF. We have

$$G = NV \cong N \circledS \rho_N(V) \quad \text{and} \quad P = N \circledS \overline{\rho_N(V)}.$$

Let $\sigma$ be a bicontinuous automorphism of $G$. Then

$$G = \sigma(N) \cdot \sigma(V) \cong \sigma(N) \circledS \rho_{\sigma(N)}(\sigma(V)).$$

Since $\rho_G(P) = \overline{I(G)}$ from Lemma 2.2, we can construct the continuous homomorphism $\Psi\colon T \to \rho_G(P)$, where

$$\Psi(\tau) = \sigma \circ \rho_G(\tau) \circ \sigma^{-1}. \tag{1}$$

It is clear that $\Psi$ is a bicontinuous isomorphism of $T$ onto a compact subgroup of $\rho_G(P)$.

From (1) we have

$$\Psi(\rho_N(v)) = \rho_G(\rho_{\sigma(N)}(\sigma(v))), \qquad v \in V. \tag{2}$$

Let $\mathrm{Cl}_P(\rho_{\sigma(N)}(\sigma(V)))$ denote the closure of $\rho_{\sigma(N)}(\sigma(V))$ in $P$. We see from (2) that the restriction of $\rho_G$ to $\mathrm{Cl}_P(\rho_{\sigma(N)}(\sigma(V)))$ is one-to-one. We also see from (1) and (2) that

$$\rho_G(\rho_{\sigma(N)}(\sigma(V))) \subset \Psi(T) \subset \mathrm{Cl}_{A(G)}(\rho_G(\rho_{\sigma(N)}(\sigma(V)))). \tag{3}$$

Since $\Psi(T)$ is compact, we see from (3) that

$$\Psi(T) = \mathrm{Cl}_{A(G)}(\rho_G(\rho_{\sigma(N)}(\sigma(V)))).$$

We now wish to show that $\mathrm{Cl}_P(\rho_{\sigma(N)}(\sigma(V)))$ is a toral group; for once we know this we can conclude that

$$\Psi(T) = \rho_G(\mathrm{Cl}_P(\rho_{\sigma(N)}(\sigma(V)))). \tag{4}$$

To this end we let $A_0(N)$ and $A_0(\sigma(N))$ denote the identity component groups of $A(N)$ and $A(\sigma(N))$, respectively. Let

$$F\colon N \circledS A_0(N) \to \sigma(N) \circledS A_0(\sigma(N))$$

be defined by $F(n, \alpha) = (\sigma(n), \alpha')$, where $\alpha'(\sigma(\bar{n})) = \sigma(\alpha(\bar{n}))$, $\bar{n} \in N$. $F$ is a bicontinuous isomorphism onto, and

$$F(P) = N \circledS \mathrm{Cl}_{A(N)}(\rho_N(V)) = \sigma(N) \circledS \mathrm{Cl}_{A(\sigma(N))}(\rho_{\sigma(N)}(\sigma(V))),$$

$$F(\mathrm{Cl}_{A(N)}(\rho_N(V))) = \mathrm{Cl}_{A(\sigma(N))}(\rho_{\sigma(N)}(\sigma(V))),$$

$$F(\rho_N(V)) = \rho_{\sigma(N)}(\sigma(V)).$$

Therefore,

$$\begin{aligned}\mathrm{Cl}_P(\rho_{\sigma(N)}(\sigma(V))) &= \mathrm{Cl}_P(F(\rho_N(V)) \\ &= F(\mathrm{Cl}_P(\rho_N(V))) = F(\mathrm{Cl}_{A(N)}(\rho_N(V))),\end{aligned}$$

where we have identified $P$ and $F(P)$. Since $T = \mathrm{Cl}_{A(N)}(\rho_N(V))$ is a toral group, we see that $\mathrm{Cl}_P(\rho_{\sigma(N)}(\sigma(V)))$ is a toral group. Therefore, (4) is true.

Now let $\Phi$ denote the inverse of the bicontinuous isomorphism $\rho_G$ from $\mathrm{Cl}_P(\rho_{\sigma(N)}(\sigma(V)))$ onto $\Psi(T)$. Define $\epsilon(\sigma)\colon P \to P$ as follows:

$$\epsilon(\sigma)(n, \tau) = \sigma(n) \cdot (\Phi \circ \Psi)(\tau). \tag{5}$$

First of all, $(\Phi \circ \Psi)(\rho_N(v)) = \Phi(\rho_G(\rho_{\sigma(N)}(\sigma(v))))$, from (2). But $\Phi(\rho_G(\rho_{\sigma(N)}(\sigma(v)))) = \rho_{\sigma(N)}(\sigma(v))$ from the definition of $\Phi$. However, $\rho_{\sigma(N)}(\sigma(v)) = \sigma(\rho_N(v))$ under the identifications $v \leftrightarrow \rho_N(v)$ and $\sigma(v) \leftrightarrow \rho_{\sigma(N)}(\sigma(v))$. Therefore, from (5) we see that $\epsilon(\sigma)(g) = \sigma(g)$ for all $g$ in $G$. Since $G$ is dense in $P$, we also see that $\epsilon(\sigma) \in A(P)$.

Now suppose that $Z(K)$ is totally disconnected (which would certainly be the case if $Z(G)$ were totally disconnected). Let $A_D(H)$ be the identity component group of the group of bicontinuous automorphisms of $H$ which leave $D$ elementwise fixed. Let $f'\colon A_D(H) \to A(P)$ be defined by $f'(\delta)(k, h)\tilde{D} = (k, \delta(h))\tilde{D}$. Then from Goto [1] we have

$$A_0(G) = I_G(K_0) \times f(A_D(L)),$$
$$A_0(P) = I_P(K_0) \times f'(A_D(H)),$$

and

$$I_G(K_0) \cong I_K(K_0) \cong I_P(K_0).$$

Now consider the following diagram:

$$\begin{array}{ccc} A_D(L) & \xrightarrow{\epsilon'} & A_D(H) \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f'} \\ f(A_D(L)) & \xrightarrow{\epsilon} & f'(A_D(H)). \end{array}$$

For $\delta \in A_D(L)$ we see from the definitions that $(f' \circ \epsilon')(\delta)$ and $(\epsilon \circ f)(\delta)$ agree on $G$. Hence, they agree on $P$. Therefore, the diagram commutes. Thus, $\epsilon$: $A_0(G) \to A(P)$ is continuous. We have now completed the proof of our main theorem.

THEOREM 2.3. *Let $G$ be a $(CA)$ locally compact connected group and let $N$ and $H$ be a closed normal connected subgroup and a closed connected subgroup of $G$, respectively, such that $G = NH$, $N \cap H = \{e\}$. Let $N = KL^*$ be a canonical decomposition of $N$, and suppose that $Z(K)$ is totally disconnected. Let $\pi$ denote the natural projection of $G$ onto $H$.*

(i) *If $\pi(Z(G))$ is discrete, then $N$ is $(CA)$.*

(ii) *If $\pi(Z(G))$ is closed and $N/Z(N)$ is homeomorphic to Euclidean space, then $N$ is $(CA)$.*

PROOF. Suppose that $N$ is non-$(CA)$. Let $N'$ be a $(CA)$ locally compact connected group such that $N \to N'$ is a dense imbedding, where $N'$ is to be constructed according to the method in Theorem 2.1. Let $\epsilon$: $A_0(N) \to A_0(N')$ be the continuous extension homomorphism constructed in Theorem 2.2. Let $\beta = \epsilon \circ \rho_N$. Then the restriction of $\beta$ to $H$ is a continuous homomorphism of $H$ into $A(N')$, and we let $G'$ denote the semidirect product of $N'$ and $H$ that is determined by $\beta$. Then $G \to G'$ is a dense imbedding.

Let $\{(n_\nu, h_\nu)\}$ be a net of central elements in $G$ converging in $G'$ to $(n', h)$.

*Case* (i). In this case we can find some $\mu$ such that for $\nu \geqslant \mu$ we have $h_\nu = h$. Therefore, $n_\nu n_\mu^{-1} = (n_\nu, h)(n_\mu, h)^{-1}$ is in $Z(G) \cap N$ for $\nu \geqslant \mu$. Since the center of $N$ is closed in $N'$ by Theorem 2.1, and since $\rho_N(H)$ keeps each $n_\nu n_\mu^{-1}$ fixed $(\nu \geqslant \mu)$, we see that $n' n_\mu^{-1}$ is in the center of $N$ and is held fixed by $\rho_N(H)$. Hence, $n' n_\mu^{-1} \in Z(G)$. Therefore, $(n', h) = z \cdot (n_\mu, h)$, $z \in Z(G)$. So the center of $G$ is closed in $G'$. Since $G$ is $(CA)$, we can appeal to Goto [2, Proposition 9] to conclude that $G = G'$. Therefore, $N$ is $(CA)$.

*Case* (ii). In this case, since $\pi(Z(G))$ is closed in $H$, there exists an element $\bar{n}$ in $N$ so that $(\bar{n}, h)$ is in $Z(G)$. Since $n'\bar{n}^{-1} = (n', h) \cdot (\bar{n}, h)^{-1}$, we see that $n'\bar{n}^{-1} \in$ (center of $G'$) $\cap N'$. Therefore, $n'\bar{n}^{-1}$ is in the center of $N'$. Since $N/Z(N)$ is homeomorphic to Euclidean space, $N$ and $N'$ have the same center by Theorem 2.1. Thus, $n'\bar{n}^{-1}$ is in the center of $N$. Therefore, since $n'\bar{n}^{-1}$ is already in the center of $G'$, it follows that $n'\bar{n}^{-1} \in Z(G)$. So $(n', h) = z \cdot (\bar{n}, h)$, $z \in Z(G)$. As in case (i) above, we can conclude that $N$ is $(CA)$.

## Bibliography

**1.** M. Goto, *On the group of automorphisms of a locally compact connected group*, Mem. Amer. Math. Soc. No. 14 (1955), 23–29. MR **16**, 997.

**2.** ——, *Dense imbeddings of locally compact connected groups*, Ann of Math. (2) **61** (1955), 154–169. MR **16**, 447.

**3.** H. Yamabe, *On the conjecture of Iwasawa and Gleason*, Ann. of Math. (2) **58** (1953), 48–54. MR **14**, 948.

**4.** ——, *A generalization of a theorem of Gleason*, Ann. of Math. (2) **58** (1953), 351–365. MR **15**, 398.

**5.** D. Zerling, *Some theorems on* ($CA$) *analytic groups*, Trans. Amer. Math. Soc. **205** (1975), 181–192.

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