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COUNTABLY GENERATED FAMILIES

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ABSTRACT. This paper discusses some interrelationships between various statements involving sets generated by rectangles, universal spaces, and real-valued measures on the continuum. Borel sets on ordinal spaces are also discussed.

Definition. Let E denote a set. If A is a subset of $E^2 = E \times E$, let $A_x = \{y|(x,y) \in A\}$, for each x in E and let $A^y = \{x|(x,y) \in A\}$, for each y in E. Let R be the family of all sets of the form $A \times B$ in E^2.

If G is a subset of 2^E, let G_0 be G and for each ordinal α, let G_α be the family of all countable unions (intersections) of sets in $\bigcup_{\gamma<\alpha} G_{\gamma'}$ if α is odd (even). Limit ordinals will be considered even. Of course, G_{ω_1} is the smallest family including G which is closed under countable unions and intersections; G_{ω_1} is the Borel lattice generated by G. It can be checked that if for each $A \in G$, $A' \in G_{\omega_1}$, then G_{ω_1} is closed under complements and G_{ω_1} is then the σ-algebra or Borel algebra generated by G.

In [1], a study is made of the Borel lattice (algebra) generated by the family R and a number of the results stated in that paper will be used here. In particular, if $|E| > c$, then the main diagonal in $E \times E$ is not in R_{ω_1}. If $|E| \leqslant \omega_1$, then $2^{E^2} = R_{\omega_1} = R_{\sigma\delta}$. (It should be noted that throughout this paper, the Axiom of Choice is assumed and cardinals are regarded as initial ordinals.)

Kunen [4] investigated the family R_{ω_1} when $|E| \leqslant c$. He showed that Martin's Axiom implies $R_{\sigma\delta} = 2^{E^2}$, if $|E| \leqslant c$.

Recently, Franklin Tall and Kenneth Kunen have constructed a model of ZFC in which Martin's Axiom fails and yet $2^{E^2} = R_{\omega_1}$, if $|E| \leqslant c$. R. Mansfield [18] and B. V. Rao [19], [20] have also studied the sets generated by rectangles and have solved some of Ulam's problems with their aid.

There are a number of interesting consequences which follow from the assumption that $2^{E^2} = R_{\omega_1}$ and from the techniques which have been used in the study of families generated by R.

Results.

THEOREM 1. *Suppose $2^{E^2} = R_{\sigma\delta}$, where $|E| = c$, and if $\lambda < c$, then $2^\lambda \leqslant c$. Then there is a subset M of $[0,1]$ such that the Banach space $B_1(M)$ consisting of the bounded real-valued functions of Baire's class 1 defined on M and under the uniform norm is universal for all Banach spaces of cardinality c.*

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PROOF. It is known that the condition $\lambda < c$ implies $2^\lambda \leqslant c$ is equivalent to $c = c^{\mathfrak{C}}$. It is known that $c = c^{\mathfrak{C}}$ implies there is a zero-dimensional compact T_2 space U of weight c such that if Y is a compact T_2 space weight $\leqslant c$, then Y is a continuous image of U [16], [17, p. 131]. If X is a Banach space and $|X| = c$, then X can be embedded in a space $C(S)$ where S is a compact T_2 space and $|C(S)| = c$. But if $|C(S)| \leqq c$, then weight of S is $\leqq c$. Thus, X can be embedded in $C(U)$. Since U has weight c, $|C(U)| \leqslant c$. Now by Theorem 4.4 of [7], there is a subset M of $[0,1]$ such that $B_1(M)$ is universal for all Banach spaces of cardinality c.

Question. Does the existence of a universal Banach space of cardinality c imply $c = c^{\mathfrak{C}}$ or that there is a compact T_2 space, U, of weight c so that every compact T_2 space of weight c is a continuous image of U?

There are a number of other consequences of the assumption that $R_2 = R_{\sigma\delta} = 2^{E^2}$ by itself or together with a cardinality condition. For example, if $2^{\aleph_1} = 2^{\aleph_0}$ and $R_2 = 2^{E^2}$, then there exists a Q-set [7, Theorem 4.5].

In [4], Kunen showed that $R_{\omega_1} = 2^{E^2}$ implied that $|E|$ is not a real-valued measurable cardinal. As Kunen points out, his argument is a variant of the known fact that a well-ordering of the real numbers is not Lebesgue measurable [11]. We shall generalize this argument as follows:

THEOREM 2. *Let $|E| = c$. For each positive integer i, statement i implies statement $i + (1)$.*

1. $2^{\aleph_0} = \aleph_1$;
2. *Martin's Axiom*;
3. $2^{E^2} = R_2 = R_{\sigma\delta}$;
4. $2^{E^2} = R_{\omega_1}$;
5. *there is a countable ordinal α such that if H is a family of c subsets of E, then there is a countable family G of subsets of E such that $H \subseteq G_\alpha$*;
6. *there is a countable ordinal α such that if H is a family of c subsets of E and each member of H has cardinality $< c$, then there is a countable family G such that $H \subseteq G_\alpha$*;
7. *if H is a family of c subsets of E and each member of H has cardinality $< c$, then there is a countable family G such that H is a subcollection of the Borel algebra generated by G*;
8. *if W is a subset of E^2, then W is the union of a subfamily G of R_{ω_1} such that the cardinality of G is not real-valued measurable*;
9. *if κ is a real-valued measurable cardinal, then $\kappa > c$.*

PROOF. For the first three implications, see Kunen [4]. As has been mentioned, $3 \nrightarrow 2$. Clearly $3 \to 4$, and in [1] it is shown that $4 \to 5$. The question "Does $4 \to 3$?" was raised in [1]. Clearly $5 \to 6$.

We shall now show that $6 \to 4$.

Let E be well ordered into an initial type; $E = \{x_1, x_2, \ldots, x_\alpha, \ldots | \alpha < c\}$. Let $A = \{(x_\alpha, x_\gamma) | \alpha < \gamma\}$ and let $B = E^2 - A$. For each y, $|A^y| < c$ and for each x, $|B_x| < c$.

Let $Z \subseteq E^2$. Then $Z = (Z \cap A) \cup (Z \cap B)$. Let $H = \{(Z \cap A)^y | y \in E\}$. Each member of H has cardinality $< c$. Thus, if 6 holds, there is a countable family of G of subsets of E and a countable ordinal α such that $H \subseteq G_\alpha$. It follows directly from Theorem 3 of [1] that $Z \cap A \in R_{\omega_1}$. Similarly, $Z \cap B \in R_{\omega_1}$. So, $6 \to 4$.

It should be noted that this argument may be used to show that if $|T| = \omega_1$, then every subset of T^2 is in the family $R_{\sigma\delta}$. This fact is proven by different means by Kunen [4].

Also, it should be pointed out that if statement 5 holds, then if H is a family of c subsets of E, there is a countable family G of subsets of E such that $H \subseteq \bigcup_{\alpha<\omega_1} G_\alpha$. This problem was raised by Ulam and Rothberger [9], [14].

Clearly, $6 \to 7$. It is unknown to the author whether $7 \to 6$.

We shall now show that $7 \to 8$.

Let A and B be the sets described above. Let $W \subseteq E^2$ and let $H = \{(A \cap W)^y | y \in E\}$. Clearly, each member of H has cardinality $< c$ and $|H| \leqslant c$. Thus, from 7, there is a G, $|G| \leqslant \omega_0$ such that $H \subseteq \bigcup_{\gamma<\omega_1} G_\gamma$.

For each $\gamma < \omega_1$, let $K_\gamma = \{(x,y) | (x,y) \in A \cap W \text{ and } (A \cap W)^y \in G_\gamma\}$. It follows from Theorem 3 of [1] that $K_\gamma \in R_\gamma$.

Thus, $W \cap A = \bigcup_{\gamma<\omega_1} K_\gamma$. There is a similar argument for $W \cap B$ and certainly $7 \to 8$ since ω_1 is not a real-valued measurable cardinal.

Finally, we show $8 \to 9$. We argue indirectly. Suppose κ, $\kappa \leqslant c$, is real-valued measurable and 8 holds. Let ω_α be the first real-valued measurable cardinal and let $S = \{\gamma | \gamma < \omega_\alpha\}$ and let $A = \{(x,y) | (x,y) \in S \times S$ and x precedes $y\}$.

It follows from statement 8 that there is a subfamily $G = \{A_\gamma\}_{\gamma<\lambda}$ of the Borel field generated by the rectangles over S such that $A = \bigcup_{\gamma<\lambda} A_\gamma$ and λ is not real-valued measurable.

Let μ be a free probability measure on ω_α which is ω_α-additive.

For each $\gamma < \lambda$, A_γ is $\mu \times \mu$-measurable. We calculate the measure of A_γ by Fubini's theorem.

$$\mu \times \mu(A_\gamma) = \int_{S\times S} \chi_{A_\gamma} d(\mu \times \mu) = \int_S \left[\int_S \xi_{A_\gamma}(x,y)\, d\mu(x) \right] d\mu(y).$$

But, for each y, $\int_S \xi_{A_\gamma}(x,y)\, d\mu(x) = \mu(A_\gamma^y) = 0$. Thus, $(\mu \times \mu)(A_\gamma) = 0$.

For each $\gamma < \lambda$, let $P_\gamma = \{x | \mu((A_\gamma)_x) > 0\}$. It follows from Fubini's theorem that each P_γ has μ-measure 0.

However, for each $x \in S$, $\mu(A_x) > 0$ and $A_x = \bigcup_{\gamma<\lambda} (A_\gamma)_x$. Thus, $\bigcup_{\gamma<\lambda} P_\gamma = S$. But, since μ is ω_α-additive, $\mu(S) = 0$. This contradiction completes the proof of the theorem.

REMARK 1. The theorem that $8 \to 9$ was also proven by E. Fisher in his thesis [2]. In fact, Fisher showed that no well-ordering of ω_α is in the ω_α-algebra generated by R. The author was unaware of this and thanks the referee for pointing this out and for making a number of other helpful comments.

REMARK 2. In the first issue of Colloquium Mathematicum, Banach showed that the continuum hypothesis implies that there is a countable family of subsets of I, the unit interval, such that Lebesgue measure cannot be extended from the Lebesgue measurable sets to a σ-algebra containing these sets. The same result holds under Martin's Axiom. Any countable family $\{E_n\}_{n=1}^{\infty}$ such that a well-ordering of I (regarded as a subset of $I \times I$) is in the σ-algebra generated by the rectangles $A_n \times A_m$ will suffice. The argument is the same as above, in view of the fact that Lebesgue measure is c-additive under Martin's Axiom [6].

As mentioned earlier, it is apparently unknown whether $R_{\omega_1} = 2^{E^2}$ implies $R_2 = R_{\sigma\delta} = 2^{E^2}$. In fact, it is apparently unknown whether there is any family of sets G such that $G_\alpha = G_{\alpha+1}$, but $G_\beta \neq G_\gamma$ for $\beta < \alpha$ and $\alpha > 3$ [3]. It is known that the Baire order of compact T_2 spaces is either 0, 1 or ω_1 (here G is the family of all closed O_δ sets) [8]. It is apparently unknown what the Borel order of a compact T_2 space may be (here G is the family of sets which are the intersection of an open set and a closed set).

We now describe the Borel subsets of the ordinal spaces $[0, \alpha)$ provided with the order topology. First, in Theorem 4, the Borel subsets of $[0, \omega_1)$ are described. This theorem was proven by M. Bhaskara Rao and K. P. S. Bhaskara Rao [21].

THEOREM 3. *Every Borel subset of $[0, \omega_1)$ can be expressed as the union of countable many sets, each of which is the intersection of an open set and a closed set.*

PROOF. Let $\mathfrak{M}$ be the σ-algebra of all subsets E of $[0, \omega_1)$ such that E or E' contains a closed unbounded subset of $[0, \omega_1)$. Clearly $\mathfrak{M}$ contains the open sets and the closed sets.

Suppose $E \in \mathfrak{M}$ and E' contains a closed unbounded set F_0. Let $\{V_\alpha\}_{\alpha \in A}$ be the set of all order components of the complement of F. Then $E \cap V_\alpha$ is countable: $E \cap V_\alpha = \{x_{\alpha n}\}_{n=1}^\infty$. Let $K_n = \{x_{\alpha n} | \alpha \in A\}$. For each n, K_n is closed in F'. Thus, $K_n = F_n \cap V$, where $F_n = \overline{K}_n$ and $V = F'$ and $E = \bigcup_{n=1}^\infty K_n$.

If $E \in \mathfrak{M}$ and E contains a closed unbounded set F_0, then as before $E - F_0 = \bigcup_{n=1}^\infty (F_n \cap V)$ and $E = F_0 \cup \bigcup_{n=1}^\infty (F_n \cap V)$ where $V = F_0'$.

Thus, $\mathfrak{M}$ is the family of all Borel subsets of $[0, \omega_1)$ and E is a Borel subset of $[0, \omega_1)$ if and only if E or E' contains a closed unbounded set.

REMARK 3. In contrast with the classical development, the smallest family containing the closed subsets of $[0, \omega_1)$ which is closed under countable unions and intersections is not the Borel algebra generated by the closed sets. In fact, let $\mathcal{K} = \{x | x$ is countable or x contains an unbounded closed set $\}$. Then $\mathcal{K}$ contains all the closed sets, $\mathcal{K}_\sigma = \mathcal{K}_\delta = \mathcal{K}$, and yet $\mathcal{K} \neq \mathfrak{M}$.

REMARK 4. It is known that the σ-algebra generated by Borel measurable rectangles in $[0, \omega_1) \times [0, \omega_1)$ does not include all Borel subsets of $[0, \omega_1) \times [0, \omega_1)$. In fact, the sets $D_1 = \{(x,y) | y > x\}$ and $D_2 = \{(x,y) | y < x\}$ are disjoint open sets which are not measurable with respect to the outer measure induced by the gauge $g(A \times B) = \mu(A) \cdot \mu(B)$, where μ is Dieudonne's measure: $\mu(E) = 1$, if E contains a closed unbounded set and $\mu(E) = 0$, otherwise.

We have

$$\mu_g^*(E) = \inf\left\{ \sum_{n=1}^{\infty} g(A_n \times B_n) \,\Big|\, \bigcup_{n=1}^{\infty} A_n \times B_n \supset E \right\}.$$

It follows that μ_g^* is $\{0,1\}$-valued. We show $\mu_g(D_1) = \mu_g(D_2) = 1$ to show that there are nonmeasurable Borel sets. If $\mu_g(D_1) = 0$, then there is a sequence $\{A_n \times B_n\}_{n=1}^\infty$ such that $D_1 \subset \bigcup_{n=1}^\infty (A_n \times B_n)$ and for each n, A_n or B_n fails to contain a closed unbounded set.

Let $A_{n_1}, A_{n_2}, \ldots$ be the sequence of all the A_n's of μ-measure 0. Let F be a closed unbounded subset of $\bigcap_{i=1}^{\infty} A'_{n_i}$. Let $x \in F$ and let $K = \{y \mid y > x\}$. Then $\{x\} \times K \subset D_1$ and $\{x\} \times K \subset \bigcup_{i=1}^{\infty}(A_{m_i} \times B_{m_i})$ where A_{m_i}'s contain unbounded closed sets. Then no B_{m_i} contains an unbounded set and yet $K \subset \bigcup_{i=1}^{\infty} B_{m_i}$. This is a contradiction.

Note. The referee points out that D_1 and D_2 are not measurable in $\mu \times \mu$ follows immediately by Fubini's theorem, as in the proof of $8 \to 9$ in Theorem 2.

THEOREM 4. *Let α be an ordinal. Every Borel subset of $[0, \alpha)$ can be expressed as the union of countably many sets, each of which is the intersection of an open set and a closed set.*

PROOF. Clearly, the theorem holds for all ordinals α, $\alpha \leqslant \omega_1$. It is also easy to show that if the theorem holds for the ordinal α, then it holds for $\alpha + 1$.

So assume α is a limit ordinal and the theorem holds for all $\beta < \alpha$. We consider two cases.

Case 1. $\mathrm{cf}(\alpha) = \alpha$.

In this case, let $\mathfrak{M} = \{E \mid E$ or E' contains a closed unbounded set and $\forall \gamma < \alpha$, $E \cap [0, \gamma)$ is Borel in $[0, \gamma)\}$.

$\mathfrak{M}$ is a σ-algebra and $\mathfrak{M}$ contains both the closed and the open subsets of $[0, \alpha)$.

Suppose $E \in \mathfrak{M}$ and $E' \supset F_0$, F_0 a closed unbounded set. Let $\{I_\gamma\}_{\gamma \in \Gamma}$ be the set of order components of $[0, \alpha) - F_0$.

Then, $E \cap I_\alpha$ is Borel in I_α. Thus, $E \cap I_\alpha = \bigcup_{n=1}^{\infty}(F_{n\alpha} \cap O_{n\alpha})$, where $F_{n\alpha}$ is closed in I_α and $O_{n\alpha}$ is open in I_α.

For each n, let $F_n = \overline{\bigcup_{\gamma \in \Gamma} F_{n\gamma}}$ and let $U_n = \bigcup_{\gamma \in \Gamma} O_{n\gamma}$. It follows that $E = \bigcup_\gamma (E \cap I_\gamma) = \bigcup_{n=1}^{\infty}(F_n \cap U_n)$.

If $E \in \mathfrak{M}$ and $E \supset F_0$, F_0 a closed unbounded set, then $(E - F_0) \supset F_0$ and we obtain $E = F_0 \cup \bigcup_{n=1}^{\infty}(F_n \cap U_n)$.

Case II. $\mathrm{cf}(\alpha) = \tau < \alpha$.

In this case fix a set $F_0 = \{\gamma_\beta\}_{\beta < \tau}$ running through α and such that F_0 is closed. Let $\{I_\sigma\}_{\sigma \in \Sigma}$ be the set of order components of $F_0{}'$. If E is Borel in $[0, \alpha)$, then $E \cap F_0$ is Borel in F_0 and $E \cap I_\sigma$ is Borel in I_σ. Since $\tau < \alpha$, $E \cap F_0 = \bigcup_{n=1}^{\infty}(\mathcal{K}_n \cap V_n)$, where $\mathcal{K}_n$ is closed in F_0 and V_n is open in F_0. Thus, $E \cap F_0 = \bigcup_{n=1}^{\infty} F_{2n} \cap U_{2n}$, F_{2n} closed in $[0, \alpha)$ and U_{2n} open in $[0, \alpha)$. For each $\sigma \in \Sigma$, $E \cap I_\sigma = \bigcup_{n=1}^{\infty}(F_{n\sigma} \cap U_{n\sigma})$. Let $F_{2n-1} = \overline{\bigcup_{\sigma \in \Sigma} F_{n\sigma}}$ and $U_{2n-1} = \bigcup_{\sigma \in \Sigma} U_{n\sigma}$. It follows that $E \cap F_0{}' = \bigcup (F_{2n-1} \cap U_{2n-1})$ and $E = \bigcup_{n=1}^{\infty}(F_n \cap U_n)$. Q.E.D.

Thus, if one considers the compact T_2 space $[0, \alpha]$, it has Borel order 1 no matter what ordinal α is.

Problem. Does this result hold for all compact scattered T_2 spaces? Is the Borel order of the other compact T_2 spaces ω_1?

In [21], it is shown that there is no nonatomic, countably additive, finite measure defined on the Borel subsets of $[0, \omega_1)$. We generalize this in the next theorem.

THEOREM 5. *If there is no real-valued measurable cardinal κ with $\kappa \leqslant \alpha$, then every countably additive finite measure defined on the Borel subsets of $[0, \alpha)$ is purely atomic.*

PROOF. Let us assume the contrary. Let us assume that α is the first ordinal for which such a measure exists and that μ is a nonatomic probability measure defined on the Borel subsets of $[0, \alpha)$.

Notice that if E is a Borel subset of $[0, \alpha)$ such that E is Borel isomorphic to some space $[0, \beta)$, with $\beta < \alpha$, then $\mu(E) = 0$. Next notice that if F is a closed cofinal subset of $[0, \alpha)$, then the open set $U = F'$ has measure zero. This can be seen as follows: First let φ be a 1-1 map from $[0, \beta)$, for some $\beta \leqq \alpha$, onto the set of order components of U. Define ν on each subset W of $[0, \beta)$ by $\nu(W) = \mu(\bigcup\{\varphi(\gamma) : \gamma \in W\})$. Then ν is a free countably additive finite measure defined on all subsets of $[0, \beta)$. Therefore $\nu([0, \beta)] = \mu(U) = 0$.

Suppose E is a Borel set which fails to contain a closed cofinal set. By the previous theorem, $E = \bigcup_{i=1}^{\infty}(F_i \cap U_i)$, where for each i, F_i is closed and U_i is open. For each i, either F_i or U_i fails to contain a closed cofinal set. If U_i does not contain such a set, then $\mu(U_i) = 0$. If F_i fails to contain such a set, then F_i is a subset of an open set not containing a closed cofinal set. Therefore, $\mu(E) = 0$.

Finally, notice that if B is a Borel set, then either B or B' contains a closed cofinal subset. But, this implies that μ is purely atomic.

REMARK. It is known that every regular Borel measure on any ordinal space (or, more generally, any compact dispersed space) is concentrated on a countable set.

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