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# ON ENDOMORPHISMS OF A SOLENOID

A. M. WILSON

ABSTRACT. Geometrically simple Bernoulli generators are constructed for certain ergodic endomorphisms of solenoids. An arbitrary ergodic solenoidal group automorphism is obtained as the limit of a sequence of such Bernoulli factors and hence, by a theorem of D. S. Ornstein, must be measure-theoretically isomorphic to a Bernoulli shift.

In his survey paper [6], B. Weiss stated that, using Y. Katznelson's methods, he can prove that every ergodic automorphism of a solenoid is isomorphic to a Bernoulli shift. The aim of this note is to give an alternative proof of this result, with a partial result in the endomorphism case.

The methods used are similar to those of L. M. Abramov [1], who used geometrically simple generating partitions in order to compute the entropy of certain solenoidal automorphisms. A comparison will show that Abramov's generators are refinements of the Bernoulli generators exhibited in §2 below.

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For brevity, a working knowledge of measure theory, ergodic theory and topological groups is assumed in what follows, apart from the following fundamental definition of a Bernoulli shift.

A measure preserving map $\phi$ from a separable measure space $(X, \mu)$ to itself will be called a (one-sided) Bernoulli shift if there is a measurable partition $P$ of $X$ (called a Bernoulli generator for $\phi$) such that

(i) $\{\phi^{-i}P\}$, $i \geqq 0$, is an independent family of partitions, and

(ii) $\bigvee_{i=0}^{\infty}\phi^{-i}P$ is the point partition of $X$.

If $\phi$ is invertible, (ii) becomes (ii)′ $\bigvee_{-\infty}^{\infty}\phi^{i}P$ is the point partition of $X$.

1. Details of solenoids are well documented (see e.g. [1], [3] and [2, Chapter VIII). The following brief characterisation will be subsequently useful.

DEFINITION 1.1. Let $G$ be a noncyclic subgroup of the discrete additive group $Q$ of rational numbers. The character group $\Sigma$ of $G$, called a (one-dimensional) solenoid, is a compact, separable, commutative topological group.

PROPOSITION 1.2. *Let* $\mathbf{a} = (a_1, a_2, \dots)$ *be a sequence of integers* $a_i \geqq 2$. *Let* $G_{\mathbf{a}}$ *be the subgroup of* $Q$ *generated by the elements* $\prod_{i=1}^{n} 1/a_i$, *for* $n \geqq 1$. *Up to isomorphism, every additive subgroup* $G$ *of* $Q$, *as in* 1.1, *can be represented as a* $G_{\mathbf{a}}$ *for some such* $\mathbf{a}$. *If* $\mathbf{a}$ *is the constant sequence on some integer* $a$, *then* $G_{\mathbf{a}}$ *is the group of a-ary rationals, denoted* $G_a$.

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The character group of $G_{\mathbf{a}}$ will be denoted $\Sigma_{\mathbf{a}}$. The character group of $G_a$ will be denoted $\Sigma_a$, called the $a$-adic solenoid. The solenoid $\Sigma_{\mathbf{a}}$ can be viewed as a subgroup of the countable product of circle groups $S$, where $\mathbf{x} = (x_0, x_1, \dots) \in \Sigma_{\mathbf{a}}$ if and only if $\forall i \geqq 0$, $x_i \in S$ and $x_{i+1}^{a_{i+1}} = x_i$.

PROPOSITION 1.3. *Each endomorphism of $G_{\mathbf{a}}$ is of the form $\psi_{m/n}$, acting as multiplication by $m/n$. By convention, $n > 0$. If $m/n \neq 0$, then $m$ and $n$ are coprime (by convention), and $G_n$ is a subgroup of $G_{\mathbf{a}}$. If $\psi_{m/n}$ is invertible, its inverse is $\psi_{n/m}$, and $G_{mn}$ is a subgroup of $G_{\mathbf{a}}$.*

The endomorphisms of $\Sigma_{\mathbf{a}}$ are in 1-1 correspondence with those of $G_{\mathbf{a}}$, so to each endomorphism $\psi_{m/n}$ the dual endomorphism $\phi_{m/n}$ of $\Sigma_{\mathbf{a}}$ is associated. The nontrivial endomorphisms $\psi_{m/n}$ (i.e. $m/n \neq 0$) of $G_{\mathbf{a}}$ are injective,so their duals acting on $\Sigma_{\mathbf{a}}$ are all surjective, Haar measure-preserving endomorphisms.

PROPOSITION 1.4. *The endomorphism $\phi_{m/n}$ of $\Sigma_{\mathbf{a}}$ is ergodic if and only if $m/n \neq 0, \pm 1$.*

2. Let $\Sigma_{\mathbf{a}}$ be a fixed solenoid, and $\phi_{m/n}$ an ergodic endomorphism of it. Up to isomorphism, $\Sigma_{\mathbf{a}}$ may be represented by $\mathbf{a} = (nb_1, nb_2, \dots)$ where each $b_i$ is a positive integer coprime to $n$. Then

$$\begin{aligned}\phi_{m/n}(x_0, x_1, \dots) &= \phi_{m/n}(x_1^{nb_1}, x_2^{nb_2}, \dots) \\ &= (x_1^{(m/n)\cdot nb_1}, x_2^{(m/n)\cdot nb_2}, \dots) = (x_1^{mb_1}, x_2^{mb_2}, \dots).\end{aligned}$$

Before exhibiting Bernoulli partitions for such endomorphisms, some notation and a general lemma must be introduced.

*Notation.* (i) Let $(S, \nu)$, $(\Sigma_{\mathbf{a}}, \mu)$ denote the circle and solenoidal groups, respectively, with normalised Haar measures.

(ii) For each nonzero integer $N$, let $S(N)$ be the partition of $S$ into $|N|$ arcs $\{S_1(N), \dots, S_{|N|}(N)\}$ where, for $1 \leqq j \leqq |N|$, $S_j(N) = \{x = \exp 2\pi i\theta\colon -j - 1/|N| \leqq \theta < j/|N|\}$.

Observe that translation by any $N$th root of unity permutes the elements of $S(N)$.

(iii) For $i \geqq 0$, let the $i$th coordinate projection $\pi_i\colon \Sigma_{\mathbf{a}} \to S$ be the measure-preserving map given by $\pi_i(x_0, x_1, \dots) = x_i$.

(iv) For nonzero integers $h$, define $\omega_h\colon S \to S$ to be the measure preserving endomorphism given by $\omega_h(x) = x^h$.

(v) We shall say that a partition $P$ of a group $X$ is regular with respect to a subgroup $K$ of $X$ if for each atom $P_i \in P$, the sets $\{xP_i\}$, $x \in K$, are disjoint and form the partition $P$ of $X$.

(vi) For each $r \geqq 1$, the product $\prod_{j=1}^{r} b_j$ will be abbreviated to $B_r$, and by convention $B_0 = 1$.

LEMMA 2.1. *Let $f$, $g$ be two surjective endomorphisms of a compact separable topological group $X$, with normalised left-invariant Haar measure $\mu$, and let $P$ be a measurable partition of $X$ such that*

(i) *the elements of* Ker $f$ *and* Ker $g$ *commute,*

(ii) $g(\text{Ker } f) = \text{Ker } f$,

(iii) *$P$ is regular with respect to* Ker $f$.

*Then for any measurable partition $Q$ of $X$, the partitions $f^{-1}Q$ and $g^{-1}P$ are independent.*

PROOF. Since $X$ is compact, $\operatorname{Ker} f$ is finite, say of order $p$. The maps $f$ and $g$ must preserve $\mu$. By (iii), $\mu P_i = 1/p$ for $1 \leqq i \leqq p$.

It follows from (iii) that left translation by any element of $\operatorname{Ker} f$ permutes the elements of $P$, and that the restriction of $f$ to each $P_i$ is a bijection onto $X$.

Let $P_i$, $Q_j$ be elements of $P$, $Q$, respectively. It must be shown that

$$\mu(g^{-1}P_i \cap f^{-1}Q_j) = \mu(g^{-1}P_i) \cdot \mu(f^{-1}Q_j) = (1/p)\mu Q_j.$$

Let $\tilde{P}$, $\tilde{Q}$ be regular partitions of $g^{-1}P_i, f^{-1}Q_j$ with respect to $\operatorname{Ker} f$, $\operatorname{Ker} g$, respectively. (The existence and measurability of such partitions is a simple consequence of Zorn's lemma.)

Then for each atom $\tilde{Q}_j$ of $\tilde{Q}$,

$$f^{-1}Q_j = \bigcup_{x \in \operatorname{Ker} f} x\tilde{Q}_j \quad \text{and} \quad \mu\tilde{Q}_j = (1/p)\mu Q_j.$$

Thus

$$\begin{aligned}\mu(g^{-1}P_i \cap f^{-1}Q_j) &= \sum_{x \in \operatorname{Ker} f} \mu(g^{-1}P_i \cap x\tilde{Q}_j) \\ &= \sum_{x \in \operatorname{Ker} f} \mu(x^{-1} \cdot g^{-1}P_i \cap \tilde{Q}_j).\end{aligned}$$

Now

$$\begin{aligned}\bigcup_{x \in \operatorname{Ker} f} x^{-1} \cdot g^{-1}P_i &= \bigcup_{x \in \operatorname{Ker} f} x\Big(\bigcup_{y \in \operatorname{Ker} g} y \cdot \tilde{P}_i\Big) \\ &= \bigcup_{y \in \operatorname{Ker} g} y\Big(\bigcup_{x \in \operatorname{Ker} f} x\tilde{P}_i\Big) \quad \text{by (i)}\end{aligned}$$

where $\tilde{P}_i$ is any atom of $\tilde{P}$. But

$$\begin{aligned}g\Big(\bigcup_{x \in \operatorname{Ker} f} x\tilde{P}_i\Big) &= \bigcup_{x \in \operatorname{Ker} f} g(x)P_i \quad \text{since } g(\tilde{P}_i) = P_i \\ &= \bigcup_{x \in \operatorname{Ker} f} xP_i \quad \text{by (ii)} \\ &= X.\end{aligned}$$

Hence $\bigcup_{x \in \operatorname{Ker} f} x\tilde{P}_i$ contains an atom of a regular partition of $X$ with respect to $\operatorname{Ker} g$, therefore $\bigcup_{x \in \operatorname{Ker} f} x^{-1} \cdot g^{-1}P_i = X$.

But $\mu(g^{-1}P_i) = \mu P_i = 1/p$, so the sets $\{x^{-1} \cdot g^{-1}P_i\}$ are disjoint as $x$ varies over $\operatorname{Ker} f$. Therefore

$$\mu(g^{-1}P_i \cap f^{-1}Q_j) = \mu\tilde{Q}_j = (1/p)\mu Q_j. \quad \text{Q.E.D.}$$

PROPOSITION 2.2. *Let $\phi_{m/n}$ be an ergodic endomorphism of $\Sigma_{\mathbf{a}}$, as above. Then for both $N = m$ and $N = n$, the partition $P = \pi_0^{-1} S(N)$ is a Bernoulli partition for $\phi_{m/n}$.*

PROOF. It will be sufficient to show that for each $r \geqq 0$, $\{\phi_{m/n}^{-i}P\}$, $0 \leqq i \leqq r$, is an independent family of partitions.

Observe

$$\bigvee_{i=0}^{r} \phi_{m/n}^{-i} P = \bigvee_{i=0}^{r} \pi_i^{-1} \omega_{B_i}^{-1} \omega_m^{-i} S(N) \tag{1}$$
$$= \pi_r^{-1} \omega_{B_r}^{-1} \left( \bigvee_{i=0}^{r} \omega_m^{-i} \omega_n^{-r+i} S(N) \right).$$

By the symmetry of (1), there is no loss of generality in assuming $N = m$. Then

$$\bigvee_{i=0}^{r} \omega_m^{-i} \omega_n^{-r+i} S(N) = \omega_n^{-r} S(m) \vee \omega_m^{-1} \bigvee_{i=1}^{r} \omega_m^{-i+1} \omega_n^{-r+i} S(m).$$

In Lemma 2.1 above, set $f = \omega_m$, $g = \omega_n^r$, $P = S(m)$ and

$$Q = \bigvee_{i=1}^{r} \omega_m^{-i+1} \omega_n^{-r+i} S(m).$$

Since $S$ is commutative, (i) is satisfied. Condition (ii) follows from the coprimeness of $m$ and $n^r$, and (iii) holds by definition of $S(m)$. Hence $\omega_n^{-r} S(m)$ and $\omega_m^{-1}(\bigvee_{i=1}^r \omega_m^{-i+1} \omega_n^{-r+i} S(m))$ are independent partitions of $S$. But $\pi_r$ and $\omega_{B_r}$ preserve measure, so $P$ and $\bigvee_{i=1}^r \phi_{m/n}^{-i} P$ are independent. The proof is completed by induction.

The rest of this section concerns the cases in which the Bernoulli partitions obtained above are generators.

PROPOSITION 2.3. *Let $\phi_{m/n}$ be an ergodic endomorphism of $\Sigma_{\mathbf{a}}$, with $|m| > n > 0$. Then $P = \pi_0^{-1} S(m)$ is a generator for $\phi_{m/n}$ if either* (i) *$\phi_{m/n}$ is not invertible and $\Sigma_{\mathbf{a}} = \Sigma_n$, or* (ii) *$\phi_{m/n}$ is invertible and $\Sigma_{\mathbf{a}} = \Sigma_{mn}$.*

PROOF. (i) Suppose $\phi_{m/n}$ is not invertible. Let $\mathbf{a} = (nb_1, nb_2, \dots)$ where each $b_i$ is coprime to $n$. Set $M = |m|$.

(1) Let $\mathbf{x}$ and $\mathbf{y}$ lie in the same atom of $\bigvee_{i=0}^{\infty} \phi_{m/n}^{-i} P$. In particular, for each $r \geqq 0$, $x_r$ and $y_r$ lie in the same atom of $\omega_{B_r}^{-1} \bigvee_{i=0}^{r} \omega_m^{-i} \omega_n^{-r+i} S(m)$ (as in 2.2(1)).

(2) It can be shown by induction that each atom of $\bigvee_{i=0}^{r} \omega_m^{-i} \omega_n^{-r+i} S(m) \vee S(n^r)$ is contained in a single arc of $\nu$-measure at most $1/M^{r+1}$.

For $u, v \in S$, let $\|u - v\|$ be the $\nu$-measure of the shorter arc joining $u$ and $v$.

Then by (1) and (2) above, for each integer $i$ with $0 \leqq i \leqq r$, there is an integer $s(r,i)$ with $0 \leqq s(r,i) < B_r M^i n^{r-i}$ such that

$$0 \leqq \|x_r - y_r\| - s(r,i)/B_r M^i n^{r-i} \leqq 1/B_r M^{r+1}. \tag{3}$$

Setting $i = 0$ in (3) gives $0 \leqq \|x_r - y_r\| - s(r,0)/B_r n^r \leqq 1/B_r M^{r+1}$.

But $x_r^{B_r n^r} = x_0$, $y_r^{B_r n^r} = y_0$, so $0 \leqq \|x_0 - y_0\| \leqq n^r/M^{r+1}$. This holds for all $r \geqq 0$, and $M > n$, hence $x_0 = y_0$.

Suppose $x_i = y_i$ for $0 \leqq i \leqq r-1$. Then $\|x_r - y_r\| = \alpha/b_r n$ for some integer $\alpha$ with $0 \leqq \alpha < b_r n$. Setting $i = r$ in (3) gives

$$0 \leqq \alpha/nb_r - s(r,r)/B_r M^r \leqq 1/B_r M^{r+1}.$$

Hence $0 \leqq B_{r-1} M^r \alpha - ns(r,r) \leqq n/M < 1$, which implies $B_{r-1} M^r \alpha = n \cdot s(r,r)$.

But since $B_{r-1}$ and $m$ are coprime to $n$, it follows that $n$ divides $\alpha$. Moreover, if $b_r = 1$, it follows that $\alpha = 0$, i.e. $x_r = y_r$.

Hence, by induction, $\mathbf{x} = \mathbf{y}$ provided $b_r = 1$, $\forall r \geqq 0$ i.e. $\Sigma_{\mathbf{a}} = \Sigma_n$.

(ii) Now suppose $\phi_{m/n}$ is invertible on $\Sigma_{\mathbf{a}}$. Without loss of generality, assume $\mathbf{a} = (mnc_1, mnc_2, \dots)$ where each $c_i$ is an integer coprime to $mn$. Let $M = |m|$ and $C_r = \prod_{j=1}^{r} c_j$, $C_0 = 1$.

Let $\mathbf{x}$ and $\mathbf{y}$ be in the same atom of $\bigvee_{-\infty}^{\infty} \phi_{m/n}^i P$. In particular, for each $r \geqq 0$, $x_r$ and $y_r$ lie in the same atom of $\omega_{C_r}^{-1} \bigvee_{-r}^{r} \omega_m^{-r-i} \omega_n^{-r+i} S(m)$ (compare 2.2(1)).

The atoms of $\bigvee_{-r}^{r} \omega_m^{-r-i} \omega_n^{-r+i} S(m) \vee S(n^{2r})$ are bounded by arcs of measure at most $1/M^{2r+1}$ (compare (2) above). Hence there are integers $t(r,i)$ for $-r \leqq i \leqq r$ with $0 \leqq t(r,i) < C_r M^{r+i} n^{r-i}$ such that

$$0 \leqq \|x_r - y_r\| - t(r,i)/C_r M^{r+i} n^{r-i} \leqq 1/C_r M^{2r+1}. \tag{4}$$

The proof proceeds in the obvious way by analogy with (i), setting $i = 0$, $i = -r$ and $i = +r$ in (4) as required.

The results of this section are now summed up in the following statement.

THEOREM 2.4. (i) *Let $\phi_{m/n}$ be an ergodic 'expansive' (i.e. $|m/n| > 1$) endomorphism of the n-adic solenoid $\Sigma_n$. Then $\phi_{m/n}$ is a one-sided Bernoulli shift.*

(ii) *The ergodic automorphism $\phi_{m/n}$ of the mn-adic solenoid $\Sigma_{mn}$ is a Bernoulli shift.*

PROOF. (i) $P = \pi_0^{-1} S(m)$ is a Bernoulli generator for $\phi_{m/n}$, by 2.2 and 2.3(i).

(ii) If $|m| > n$, then $P = \pi_0^{-1} S(m)$ is a Bernoulli generator for $\phi_{m/n}$, by 2.2 and 2.3(ii). If $|m| < n$, then $Q = \pi_0^{-1} S(n)$ is a Bernoulli generator for $\phi_{n/m}$ by the same argument. But $\phi_{m/n}$ is the inverse of $\phi_{n/m}$, so they share Bernoulli generators.

3. Let $\phi_{m/n}$ now be a fixed but arbitrary ergodic automorphism of a solenoid $\Sigma_{\mathbf{a}}$, and let $M = \max(|m|, n)$. Assume $\mathbf{a} = (a_1, a_2, \dots)$ and set $A_r = \prod_{i=1}^{n} a_i$. For $r > 0$, let $P_r = \phi_{A_r} \pi_0^{-1} S(M) = \pi_r^{-1} S(M)$. Now $\phi_{A_r}$ is a $\mu$-preserving endomorphism of $\Sigma_{\mathbf{a}}$, so by 2.2, $P_r$ is a Bernoulli partition for $\phi_{m/n}$. Let $S_r$ be the $\sigma$-algebra generated by the partition $\bigvee_{-\infty}^{\infty} \phi_{m/n}^i P_r$. It follows from the proof of 2.3 (concerning $P_0$, and the definition of $P_r$, that $S_r$ separates points of $\Sigma_{\mathbf{a}}$ having distinct $r$th coordinates. Hence, $\{S_r\}$, $r \geqq 0$, is a sequence of $\sigma$-algebras increasing to the full $\sigma$-algebra on $\Sigma_{\mathbf{a}}$, and $\phi_{m/n}$ restricted to each $S_r$ is a Bernoulli shift (with Bernoulli generator $P_r$). Hence, the automorphism $\phi_{m/n}$ of $\Sigma_{\mathbf{a}}$ is isomorphic to a generalised Bernoulli shift in the sense of D. S. Ornstein [4], and by his theorem is thus isomorphic to a Bernoulli shift with entropy Log $M$.

REMARK. Since Ornstein's theorem only applies in the invertible case, it cannot be deduced by the above method that an arbitrary expansive endomorphism of a solenoid is isomorphic to a one-sided Bernoulli shift.

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Department of Statistics, The University, Manchester M13 9PL, England