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ON DIRECT SUMS OF REFLEXIVE OPERATORS

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ABSTRACT. Let A_1 and A_2 be reflexive operators on a Hilbert space H. If A_2 is algebraic then $A_1 \oplus A_2$ is reflexive.

1. **Introduction.** Let H be a complex Hilbert space and $B(H)$ the algebra of all bounded linear operators on H. For $A \in B(H)$, Lat A will denote the set of closed subspaces of H invariant under A. If $\mathcal{L}$ is a family of closed subspaces in H, then Alg $\mathcal{L}$ will denote the (weakly closed) algebra of operators which leave invariant every member of $\mathcal{L}$. $\mathcal{U}(A)$ will denote the weak closure of the algebra of polynomials in A. $(A)'$ will denote the commutant of A.

DEFINITION. A is reflexive if $\mathcal{U}(A) = \text{Alg Lat } A$. Reflexive operators have been studied by various authors ([1], [2], [4]) and the following question was raised in [3]: If A_1 and A_2 are reflexive, is $A_1 \oplus A_2$ reflexive?

In this note it is shown that if A_2 is algebraic then $A_1 \oplus A_2$ is reflexive.

2. **Preliminaries.** $H^{(k)}$ will denote the direct sum of k copies of H, and for $A \in B(H)$, $A^{(k)}$ will be the k-fold copy of A acting on $H^{(k)}$, i.e. if $\langle x_1, \ldots, x_k \rangle \in H^{(k)}$, then $A^{(k)} \langle x_1, \ldots, x_k \rangle = \langle Ax_1, \ldots, Ax_k \rangle$.

LEMMA 1. *If for every positive integer* $n \geqq 1$, Lat $A^{(n)} \subseteq$ Lat $B^{(n)}$, *then* $B \in \mathcal{U}(A)$.

This lemma is quite well known and is a standard tool used in the study of reflexive operators.

3. **Algebraic operators.** In this section we obtain some results for algebraic operators. These will be used in the proof of the main theorem.

THEOREM 1. *Let A be an algebraic operator. Then* $\mathcal{U}(A) = (\text{Alg})A) \cap (A)'$.

PROOF. Suppose $B \in (\text{Alg Lat } A) \cap (A)'$. We will show that for all positive integers n, Lat $A^{(n)} \subseteq$ Lat $B^{(n)}$. The result will then follow by Lemma 1.

Suppose $\mathcal{M} \in$ Lat $A^{(n)}$. Since every invariant subspace of $A^{(n)}$ is a span of cyclic subspaces, we can assume $\mathcal{M}$ is a cyclic subspace. Since $A^{(n)}$ is algebraic, $\mathcal{M}$ is finite dimensional.

For $1 \leqq i \leqq n$, let $\pi_i(\mathcal{M})$ be the projection of $\mathcal{M}$ on the ith coordinate space and let $\pi(\mathcal{M}) = \bigvee_{i=1}^{m} \pi_i(\mathcal{M})$. Then $\pi(\mathcal{M})$ is finite dimensional and is clearly in Lat A (since $\pi_i(\mathcal{M})$ is, for each i). Therefore, $\pi(\mathcal{M}) \in$ Lat B.

Denote the restrictions of A and B to $\pi(\mathcal{M})$ by $\hat{A}$ and $\hat{B}$ respectively. Then

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$\hat{B} \in (\text{Alg Lat } \hat{A}) \cap (\hat{A})'$. Thus, by [2], there is a polynomial p such that $Bx = p(A)x$ for all $x \in \pi(\mathfrak{M})$.

Thus if $\langle x_1, \ldots, x_n \rangle$ is an arbitrary vector in $\mathfrak{M}$, $x_i \in \pi(\mathfrak{M})$ for $1 \leqq i \leqq n$. Then $Bx_i = p(A)x_i$ and $B^{(n)}\langle x_1, \ldots, x_n \rangle = \langle p(A)x_1, \ldots, p(A)x_n \rangle$. This completes the proof.

COROLLARY 1. *Let A and B be reflexive algebraic operators. Then $A \oplus B$ is reflexive.*

PROOF. Suppose $\text{Lat } A \oplus B \subseteq \text{Lat } T$. Since $\{\langle x, 0 \rangle : x \in H\}$ and $\{\langle 0, x \rangle : x \in H\}$ are in $\text{Lat } A \oplus B$, it follows that $T = E \oplus F$ with $\text{Lat } A \subset \text{Lat } E$ and $\text{Lat } B \subset \text{Lat } F$. Since A and B are reflexive, $AE = EA$ and $FB = BF$.

Now $A \oplus B$ is algebraic and $E \oplus F \in [\text{Alg Lat}(A \oplus B)] \cap (A \oplus B)'$, so we can apply Theorem 1.

4. **The main result.** In this section we show that if A_1 is algebraic reflexive and A_2 is reflexive then $A_1 \oplus A_2$ is reflexive.

We will first prove a special case and the above result will easily follow.

THEOREM 2. *Suppose A is nilpotent reflexive and B is reflexive. Then $A \oplus B$ is reflexive.*

PROOF. Suppose $T \in \text{Alg Lat}(A \oplus B)$. The argument used in Corollary 1 shows that $T = E \oplus F$ with $E \in \mathfrak{A}(A)$ and $F \in \mathfrak{A}(B)$. Thus $E \oplus F \in (A \oplus B)'$ and $\text{Lat } A \oplus B \subseteq \text{Lat } E \oplus F$.

It will be shown that $\text{Lat}(A \oplus B)^{(n)} \subseteq \text{Lat}(E \oplus F)^{(n)}$. It suffices to show this for $n = 2$. For then the same hypotheses apply to $(A \oplus B)^{(2)}$. We can then apply the same argument to show that $\text{Lat}(A \oplus B)^{(4)} \subseteq \text{Lat}(E \oplus F)^{(4)}$. If we continue this process we obtain that $\text{Lat}(A \oplus B)^{(2^n)} \subseteq \text{Lat}(E \oplus F)^{(2^n)}$. But this implies $\text{Lat}(A \oplus B)^{(k)} \subseteq \text{Lat}(E \oplus F)^{(k)}$ for all integers $k \geqq 1$.

We identify $\text{Lat}(A \oplus B)^{(2)}$ with $\text{Lat}(A^{(2)} \oplus B^{(2)})$, and will show

$$\text{Lat}(A^{(2)} \oplus B^{(2)}) \subseteq \text{Lat}(E^{(2)} \oplus F^{(2)}).$$

Suppose $\mathfrak{M} \in \text{Lat}(A^{(2)} \oplus B^{(2)})$. We consider two cases:

Case (i). $\mathfrak{M}$ does not contain a vector of the form $\langle 0, y \rangle$. Here 0 is the zero vector in $H^{(2)}$ and $y \in H^{(2)}$. Then (by a well-known argument) $\mathfrak{M}$ is the graph of some closed operator T with domain $\mathfrak{D}(T)$ in $H^{(2)}$; $\mathfrak{M} = \{\langle x, Tx \rangle : x \in \mathfrak{D}(T)\}$. Then $\mathfrak{M} \in \text{Lat}(A^{(2)} \oplus B^{(2)})$ implies $B^{(2)}Tx = TA^{(2)}x$, $x \in \mathfrak{D}(T)$. Suppose the index of A is n. Then $A^{(2)n} = 0$. This implies that $B^{(2)n}Tx = 0$ for $x \in \mathfrak{D}(T)$. If $\mathfrak{N}(L)$ is the null space of the operator L, then $\mathfrak{M} \subseteq \mathfrak{N}(A^{(2)n} \oplus B^{(2)n}) = \{\mathfrak{N}([A \oplus B]^n)\}^{(2)}$.

Denote the restrictions of $A \oplus B$ and $E \oplus F$ to $\mathfrak{N}([A \oplus B]^n)$ by S_1 and S_2 respectively. Then S_1 is nilpotent, and $S_2 \in (\text{Alg Lat } S_1) \cap (S_1)'$. By Theorem 1, $S_2 \in \mathfrak{A}(S_1)$. This clearly implies that $\mathfrak{M} \in \text{Lat}(E^{(2)} \oplus F^{(2)})$.

Case (ii). $\mathfrak{M}$ contains a vector of the form $\langle 0, y \rangle$ with $y \neq 0$. Let $\mathfrak{N} = \{y : \langle 0, y \rangle \in \mathfrak{M}\}$ and let $\mathfrak{M}'$ denote $\mathfrak{M} \ominus (\{0\} \oplus \mathfrak{N})$.

Then $\mathfrak{M}'$ is the graph of some closed operator T; $\mathfrak{M}' = \{\langle x, Tx \rangle : x \in \mathfrak{D}(T)\}$. Also,

$$\begin{aligned} A^{(2)} \oplus B^{(2)}\langle x, Tx\rangle &= \langle A^{(2)}x, B^{(2)}Tx\rangle \\ &= \langle A^{(2)}x, TA^{(2)}x\rangle + \langle 0, (B^{(2)}T - TA^{(2)})x\rangle, \end{aligned}$$

where the last element is in $\{0\} \oplus \mathfrak{N}$.

We can assume that $\mathfrak{M}$ is a cyclic invariant subspace of $A^{(2)} \oplus B^{(2)}$. Thus there exist vectors x_1, x_2, y_1, y_2 such that

$$\mathfrak{M} = \bigvee_{i=0}^{\infty} \langle A^i x_1, A^i x_2, B^i y_1, B^i y_2\rangle.$$

Then if the index of A on $\mathfrak{D}(T)$ is n, $\mathfrak{N} = \bigvee_{i=n}^{\infty}\langle B^i y_1, B^i y_2\rangle$ and $\mathfrak{D}(T)$ and $\mathfrak{R}(T)$ are finite dimensional. Let P denote the projection on $\mathfrak{R}(T)$. Since $\mathfrak{N} \in \operatorname{Lat} B^{(2)}$, it follows that $\mathfrak{M}' \in \operatorname{Lat}(A^{(2)} \oplus PB^{(2)}P)$. This implies that $TA^{(2)} = PB^{(2)}PT$ and therefore $PB^{(2)}P$ is nilpotent of index n.

Since $F \in \mathfrak{U}(B)$ and $\mathfrak{R}(T) \oplus \mathfrak{N} \in \operatorname{Lat} B^{(2)}$, it is easy to see that $PF^{(2)}P \in \mathfrak{U}(PB^{(2)}P)$. Thus there exist polynomials q and r, both of degree less than n, such that $E^{(2)} = q(A^{(2)})$ and $PF^{(2)}P = r(PB^{(2)}P)$.

To complete the proof it is enough to show that $\mathfrak{M}' \in \operatorname{Lat}(E^{(2)} \oplus PF^{(2)}P)$. This is equivalent to showing that $r(PB^{(2)}P) = q(PB^{(2)}P)$. Let

$$\mathfrak{M}_1 = \bigvee_{i=0}^{\infty}\langle A^i x_1, B^i y_1\rangle, \qquad \mathfrak{M}_2 = \bigvee_{i=0}^{\infty}\langle A^i x_2, B^i y_2\rangle.$$

Since we identify $\mathfrak{M}$ with the subspace $\bigvee_{i=0}^{\infty}\langle A^i x_1, B^i y_1, A^i x_2, B^i y_2\rangle$, then $\mathfrak{M} \subset \mathfrak{M}_1 \oplus \mathfrak{M}_2$. Since A is nilpotent, $\mathfrak{M}_i = \mathfrak{M}_i' \oplus (\{0\} \oplus \mathfrak{N}_i)$ where $\mathfrak{N}_i = \bigvee_{k=n}^{\infty} B^k y_i$, and $\mathfrak{M}_i' = \{\langle x, S_i x\rangle : x \in \mathfrak{D}(S_i)\}$. Then $\mathfrak{D}(S_i) = \bigvee_{k=0}^{n-1} A^k x_i$ and therefore $\mathfrak{D}(T) \subseteq \mathfrak{D}(S_1) \oplus \mathfrak{D}(S_2)$. Also $\mathfrak{N} \subseteq \mathfrak{N}_1 \oplus \mathfrak{N}_2$. By taking orthogonal complements, we obtain that $\mathfrak{R}(S_1) \oplus \mathfrak{R}(S_2) \subseteq \mathfrak{R}(T)$.

Now $\mathfrak{M}_i \in \operatorname{Lat} A \oplus B$ for each i. If Q_i is the projection on $\mathfrak{R}(S_i)$, it follows that $S_i A = Q_i B Q_i S_i$, $i = 1, 2$. Thus $Q_i B Q_i$ is nilpotent and its index on $\mathfrak{R}(S_i)$ is the index of A on $\mathfrak{D}(S_i)$. It easily follows that the index of $(Q_1 \oplus Q_2)B^{(2)}(Q_1 \oplus Q_2)$ on $\mathfrak{R}(S_1) \oplus \mathfrak{R}(S_2)$ is n. Recall that this is the index of $PB^{(2)}P$ on $\mathfrak{R}(T)$.

Since $\mathfrak{M}_i \in \operatorname{Lat} A \oplus B \subseteq \operatorname{Lat} E \oplus F$, and $E = q(A)$, it follows that $Q_i F Q_i = q(Q_i B Q_i)$. Thus

$$[Q_1 \oplus Q_2]F^{(2)}[Q_1 \oplus Q_2] = q([Q_1 \oplus Q_2]B^{(2)}[Q_1 \oplus Q_2]).$$

Now $\mathfrak{R}(S_1) \oplus \mathfrak{R}(S_2)$ is an invariant subspace of $PB^{(2)}P$. Thus on $\mathfrak{R}(T) = [\mathfrak{R}(S_1) \oplus \mathfrak{R}(S_2)] \oplus \mathfrak{L}$,

$$PB^{(2)}P = \begin{pmatrix} B_1 & X \\ 0 & B_2 \end{pmatrix}$$

and

$$PF^{(2)}P = r(PB^{(2)}P) = \begin{pmatrix} r(B_1) & X' \\ 0 & r(B_2) \end{pmatrix}.$$

Thus $r(B_1) = q(B_1)$. If $t = r - q$ then $t(B_1) = 0$. But the degrees of r and q

are less than n and B_1 is nilpotent of index n. This implies that $r = q$. This completes the proof.

THEOREM 3. *Suppose A is algebraic. If A and B are reflexive, then so is $A \oplus B$.*

PROOF. Since A is algebraic, A is similar to an operator $A' = N_1 \oplus N_2 \oplus \cdots \oplus N_k$ where each N_i is a translated nilpotent operator with all translating scalars distinct. By the argument used in [2] (which holds equally well in the infinite-dimensional case) A' is reflexive if and only if N_i is reflexive for $1 \leqq i \leqq k$. Since reflexivity is preserved under similarities the result follows by induction.

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