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ISOLATED SINGULARITIES OF QUADRATIC DIFFERENTIALS ARISING FROM A MODULE PROBLEM

JEFFREY CLAYTON WIENER

ABSTRACT. If $R \subset S$ are Riemann surfaces, we will say that $z_0 \in S - R$ is an isolated point boundary component of R if there exists a neighborhood U of z_0 in S such that $U - \{z_0\} \subset R$. We prove that the quadratic differential $Q(z)\,dz^2$ obtained by solving the module problem $P(a_1, \ldots, a_k)$ applied to a free family of homotopy classes on R can be extended to $z_0 \in S$ so that either $Q(z)$ is regular at z_0 or $Q(z)$ has a simple pole at z_0.

Introduction. Let R denote an open Riemann surface. A free family of homotopy classes on R is a set of nontrivial homotopy classes H_i, $i = 1, \ldots, k$, such that all classes H_i are distinct, no class H_i is the class of a point cycle [3, p. 40], and there exist k Jordan curves $h_i \in H_i$, $i = 1, \ldots, k$, no two of which have a common point. Take k nonnegative values $a_1, \ldots, a_k$ so that $a_1 + \cdots + a_k \neq 0$. Let $P(a_1, \ldots, a_k)$ denote the module problem associated with a free family of k homotopy classes on R (see Definition 2). J. A. Jenkins and N. Suita [4] proved that for the module problem $P(a_1, \ldots, a_k)$ there exists a quadratic differential $Q(z)\,dz^2$ on R with the property that the trajectories of $Q(z)\,dz^2$ (see [4, Theorem 1] or Theorem 1 for a precise statement of this result) divide R into a family of doubly-connected domains D_i, $i = 1, \ldots, k$, such that

$$\iint_R |Q(u)|\, dA_u = \sum_{i=1}^{k} a_i^2 M_i,$$

where M_i is the module of D_i.

Let $S \subset R$ denote a Riemann surface. Suppose that z_0 is a point of $S - R$ such that there exists a neighborhood U of z_0 on S with $U - \{z_0\} \subset R$. We will show that $Q(z)\,dz^2$ can be extended to $z_0 \in S$ so that either $Q(z)$ is regular at z_0 or $Q(z)$ has a simple pole at z_0.

1. **Prelimaries.** The following three definitions appear in [4, pp. 106–110].

DEFINITION 1. By a free family of homotopy classes H on a Riemann surface R we mean a set of homotopy classes H_i, $i = 1, \ldots, k$, such that:

(1) all classes H_i are distinct and nontrivial;

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(2) there exist Jordan curves $h_i \in H_i$, $i = 1, \ldots, k$, no two of which have a common point;

(3) no class H_i is the class of a point cycle [3, p. 40].

Let R denote a Riemann surface and H a free family of k homotopy classes $H_1, \ldots, H_k$ on R. Let $a_1, \ldots, a_k$ denote k nonnegative numbers, not all zero. We now define what we mean by the module problem $P(a_1, \ldots, a_k)$ on R associated with H.

DEFINITION 2. Let $\rho(u)\,|du|$ denote a conformally invariant metric (linear density) on R such that, for any rectifiable arc c in the coordinate neighborhood of a local (uniformizing) parameter, $\int_c \rho(u)\,|du|$ exists. Suppose that $\iint_R \rho^2\,dA_u$ exists and is finite. If for each $i = 1, \ldots, k$ and every locally rectifiable $h(i) \in H_i$, $\int_{h(i)} \rho(u)\,|du| \geqslant a_i$, then the metric $\rho(u)\,|du|$ will be called admissible for the module problem $P(a_1, \ldots, a_k)$. The extremal problem consists of finding the greatest lower bound $M(a_1, \ldots, a_k)$ of $\iint_R \rho^2\,dA_u$ as $\rho(u)\,|du|$ ranges over all admissible metrics. If the greatest lower bound is obtained for a particular metric the latter is called an extremal metric for the problem.

An extremal metric is uniquely determined up to sets of measure zero. For this reason, if an extremal metric exists, it is called the extremal metric for the module problem.

DEFINITION 3. By an admissible family of doubly-connected domains associated with a free family of homotopy classes H_i, $i = 1, \ldots, k$, on a Riemann surface R, we mean a finite set of domains $D_{i(j)}$, $j = 1, \ldots, l$, $l \leqslant k$, $i(j) < i(j')$ for $j < j'$, $1 \leqslant i(j) \leqslant k$, on R such that:

(1) no two domains $D_{i(j)}$, $j = 1, \ldots, l$, have a common point;

(2) a simple closed curve in $D_{i(j)}$ separating the boundary components of $D_{i(j)}$ belongs to $H_{i(j)}$ when given the appropriate sense.

For a class H_i to which no doubly-connected domain is assigned we may say that the corresponding domain D_i is degenerate and we assign to D_i the module zero. With this interpretation we may assume that the index in Definition 3 is always $i = 1, \ldots, k$.

A bordered Riemann surface $\overline{R}$ is a one dimensional complex manifold with boundary $\partial\overline{R}$. A finite Riemann surface is a compact bordered Riemann surface.

The definition of a free family of homotopy classes on a finite bordered Riemann surface $\overline{R}$ [2, pp. 440–441] allows homotopy classes of arcs with end points on $\partial\overline{R}$. In the course of such a homotopy the end points of the arc in question are permitted to move on their respective (not necessarily distinct) boundary components.

DEFINITION 4. By a free family of homotopy classes on a finite bordered Riemann surface we mean a set of homotopy classes H_i, $i = 1, \ldots, k$, where H_i, $i = 1, \ldots, j$, say, are homotopy classes of simple closed curves and H_i, $i = j + 1, \ldots, k$, are homotopy classes of arcs joining boundary components (either set may be empty) such that:

(1) all classes H_i, $i = 1, \ldots, k$, are distinct and nontrivial;

(2) no class H_i, $i = 1, \ldots, k$, consists of closed curves homotopic to a point of $\overline{R}$;

(3) there exist simple closed curves $h_i \in H_i$, $i = 1, \ldots, j$, and arcs $h_i \in H_i$,

$i = j + 1, \ldots, k$, on $\overline{R}$ no two of which have a common point.

We shall say that a quadrangle D on $\overline{R}$ is associated with the homotopy class H_i, $i = j + 1, \ldots, k$, if a pair of opposite sides of D lie, respectively, on $\partial\overline{R}$ joined by arcs in H_i and if the class of arcs lying in D and joining these sides is contained in H_i.

By an admissible family of domains D associated with a free family of homotopy classes H_i, $i = 1, \ldots, k$, on R we mean a family of domains (quadrangles and doubly-connected domains) each associated with a class H_i and not more than one associated with any such class. Furthermore, we require that no two domains in the family have a point in common. Finally, for a class H_i, $i = 1, \ldots, k$, to which no domain has been associated in the family D, we say that the corresponding domain is degenerate and has module zero.

The definitions of $P(a_1, \ldots, a_k)$ and $M(a_1, \ldots, a_k)$ are the same as above.

2. **Known results**. The following theorem was proven by Jenkins and Suita [4, Theorem 1].

THEOREM 1. *Let R denote a Riemann surface and H_i, $i = 1, \ldots, k$, a free family of homotopy classes on R. Then for the module problem $P(a_1, \ldots, a_k)$ there exists an extremal metric of the form $|Q(u)|^{1/2}|du|$ where $Q(u)\,du^2$ is a regular quadratic differential on R.*

For an admissible family of domains D_i, $i = 1, \ldots, k$, associated with the free family of homotopy classes,

$$\sum_{i=1}^{k} a_i^2 M_i \leqslant M(a_1, \ldots, a_k),$$

where M_i is the module of D_i.

The next theorem considers a free family of homotopy classes on a finite Riemann surface [2, Theorem 1].

THEOREM 2. *Let $\overline{R}$ denote a finite bordered Riemann surface on which are given a finite number (possibly zero) of distinguished points. Let R' be obtained from $\overline{R}$ by deleting these distinguished points. Let H_i, $i = 1, \ldots, k$, be a free family of homotopy classes on R'. Then for the module problem $P(a_1, \ldots, a_k)$ there exists an extremal metric $|Q(u)|^{1/2}|du|$ where $Q(u)\,du^2$ is a quadratic differential on $\overline{R}$ regular apart from possible simple poles at the distinguished points.*

Provided that neither is $\overline{R}$ a closed Riemann surface of genus 1 nor is $\overline{R} - \partial\overline{R}$ a doubly-connected domain (in either case without any distinguished points), the trajectories of $Q(u)\,du^2$ which have limiting end points at its finite critical points, together with those which pass through distinguished points, divide $\overline{R}$ into an admissible family D of domains D_i, $i = 1, \ldots, k$, associated with the given free family of homotopy classes H_i. If M_i is the module of D_i, then

$$\iint_{\overline{R}} |Q(z)|\, dA_z = M(a_1, \ldots, a_k) = \sum_{i=1}^{k} a_i^2 M_i.$$

3. **Result**. We now define an isolated point boundary component of an open Riemann surface.

DEFINITION. Let R be an open Riemann surface. Suppose that there is a Riemann surface $S \subset R$ and a point $z_0 \in S - R$ such that there exists a neighborhood U of z_0 in S so that $U - \{z_0\} \subset R$. Then z_0 will be called an isolated point boundary of R.

Let R denote an open Riemann surface and suppose that z_0 is an isolated point boundary component of R. Using Theorem 1 we conclude that there exists an extremal metric on R of the form $|Q(u)|^{1/2}|du|$, where $Q(u)\,du^2$ is a regular quadratic differential on R, associated with the module problem $P(a_1, \ldots, a_k)$ for a free family of homotopy classes H_i, $i = 1, \ldots, k$, on R.

For suitable definition of local uniformizing parameters at z_0, $S = R \cup \{z_0\}$ is a Riemann surface. We will prove that $Q(u)$ can be extended to $z_0 \in S$ so that either $Q(u)$ is regular at z_0 on S or $Q(u)$ has a simple pole at z_0 on S.

Let $\{R_n\}$ denote a canonical exhaustion of S with $z_0 \in R_n - \partial R_n$ for each n. We may assume that H_i, $i = 1, \ldots, k$, determines a free family of homotopy classes H_{in}, $i = 1, \ldots, k$, on each $R_n - (\{z_0\} \cup \partial R_n)$. Using Theorem 2, we conclude that there exists a quadratic differential $Q_n(z)\,dz^2$ with at worst a simple pole at z_0 such that $|Q_n(z)|^{1/2}|dz|$ provides the extremal metric for the module problem $P(a_1, \ldots, a_k)$ on R_n.

Let the corresponding decomposition of R_n into an admissible family of domains associated with H_{in}, $i = 1, \ldots, k$, be given by D_{in}, $i = 1, \ldots, k$, and let M_{in} denote the module of D_{in}. It is known that for fixed i, M_{in} is uniformly bounded by π times the reciprocal of the Huber module on R for the class H_i (see [3, pp. 42–43]), the latter quantity being finite since H_i is nontrivial and not the class of a point cycle.

Let D_i, $i = 1, \ldots, k$, denote any admissible family of domains on R associated with H_i, $i = 1, \ldots, k$, and let M_i denote the module of D_i. Using Theorem 1,

$$\sum_{i=1}^{k} a_i^2 M_i \leqslant M(a_1, \ldots, a_k) < \infty.$$

Since H_{in}, $i = 1, \ldots, k$, is a free family of homotopy classes determined by H_i on $R_n - (\partial R_n \cup \{z_0\})$ and $R_{n+1} \supset R_n$, we can think of each curve in H_{in} as a curve in H_{in+1}, which in turn is a curve in H_i. With this interpretation ($H_{in} \subset H_{in+1}$ and $H_{in} \uparrow H_i$ for fixed i) it should be clear that $M_{in} \to M_i$ as $n \to \infty$. Using Theorem 2,

$$\iint_{R_n} |Q_n(z)|\, dA_z = \sum_{i=1}^{k} a_i^2 M_{in} \to \sum_{i=1}^{k} a_i^2 M_i < \infty.$$

So $\{\iint_{R_n} |Q_n(z)|\, dA_z\}$ is uniformly bounded.

Let (U, ϕ) denote a parametric disc on S containing z_0, that is, U is an open subset of S containing z_0 and ϕ is a homeomorphism of U onto $\{|z| < 1\}$ such that $\phi(z_0) = 0$. In U, $Q_n(z) = h_n(z) + \beta(n)/z$, where $h_n(z)$ is regular in U and $\beta(n)$ is a complex number. In fact, we can assume that $h_n(z)$ is regular on the closure of U, denoted by Cl U.

We will show that a subsequence of $\{\beta(n)\}$ converges. Note that

$$\beta(n) = \frac{1}{2\pi i} \int_{|z|=r} Q_n(z)\, dz, \qquad r < 1.$$

So $|\beta(n)| \leqslant (1/2\pi)\int_0^{2\pi} |Q_n(re^{i\theta})| r\, d\theta$. Integrating from $r = 0$ to $r = 1$,

$$\begin{aligned} |\beta(n)| &\leqslant \frac{1}{2\pi} \iint_{\mathrm{Cl}\, U} |Q_n(re^{i\theta})| r\, dr\, d\theta \\ &= \frac{1}{2\pi} \iint_{\mathrm{Cl}\, U} |Q_n(z)|\, dA_z \leqslant M(a_1, \ldots, a_k)/2\pi < \infty. \end{aligned}$$

So there exists a subsequence of $\{\beta(n)\}$ which converges to β, $|\beta| < \infty$. We continue to designate this subsequence by $\{\beta(n)\}$. Now

$$\begin{aligned} \iint_{\mathrm{Cl}\, U} |h_n(z)|\, dA_z &= \iint_{\mathrm{Cl}\, U} |Q_n(z) - \beta(n)/z|\, dA_z \\ &\leqslant \iint_{\mathrm{Cl}\, U} |Q_n(z)|\, dA_z + \int_0^{2\pi} \int_0^1 |\beta(n)|\, dr\, d\theta \\ &= \iint_{\mathrm{Cl}\, U} |Q_n(z)|\, dA_z + 2\pi |\beta(n)|, \end{aligned}$$

which is uniformly bounded on Cl U. As was done in [4, Lemma 1], we can conclude that $|Q_n(z) - \beta(n)/z|$ is uniformly bounded on compact subsets of U.

By Weierstrass's theorem, there exists a subsequence $\{Q_{n(s)}(z)\}$ of $\{Q_n(z)\}$ such that $Q_{n(s)}(z) - \beta(n(s))/z$ converges to a regular function $f(z)$ on U. So $Q_{n(s)}(z) \to f(z) + \beta/z = Q(z)$ on U. Thus, either $Q(z)$ is regular in U ($\beta = 0$) or $Q(z)$ is regular in $U - \{z_0\}$ and has a simple pole at $z_0 \in S$.

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Department of Mathematics, Georgia Institute of Technology, Atlanta, Georgia 30332