# THE NUMERICAL RANGE OF AN UNBOUNDED OPERATOR 

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#### Abstract

The numerical range of an unbounded linear operator on a complex Banach space is the whole complex plane.


Let $X$ denote a Banach space over $\mathbf{C}, X^{\prime}$ the dual space of $X, S$ $=\{x \in X:\|x\|=1\}$, and let $T$ be an unbounded operator defined on the whole of $X$. Given $x \in S$, let $V(T, x)=\left\{f(T x): f \in X^{\prime},\|f\|=f(x)=1\right\}$. The numerical range $V(T)$ is defined by

$$
V(T)=\cup\{V(T, x): x \in S\}
$$

J. R. Giles and G. Joseph [2] prove that the semi-inner-product numerical range $W(T)$ has a certain density property, and B. Bollobas and S. Eldridge (preprint) prove that $W(T)$ is dense in $\mathbf{C}$. These imply the corresponding results for $V(T)$.
Theorem. $V(T)=\mathbf{C}$.
We use the following slight extension of Theorem 1 of [1].
Lemma. Let $x, y \in X$, and operator $R$ be defined on $\lim (x, y)$. Suppose that $\|x+R y\|<\|x\|-(8\|R x\|\|y\|)^{1 / 2}$. Then $\cup\{V(R, z): z \in S \cap \lim (x, y)\}$ contains 0 as an interior point.

Proof. There is a continuous linear operator $R_{1}$ on $X$ such that $R=R_{1}$ on $\lim (x, y)$. The proof in [1] shows that 0 is an interior point of $\cup\left\{V\left(R_{1}, z\right): z\right.$ $\in S \cap \lim (x, y)\}$ which gives the result.
Proof of Theorem. As in [2], there is a sequence ( $x_{n}$ ) in $X$ such that $x_{n} \rightarrow 0$ and $T x_{n} \rightarrow-x \neq 0$. Choose $x_{n}$ such that

$$
\left\|x+T x_{n}\right\|<\|x\|-\left(8\|T x\|\left\|x_{n}\right\|\right)^{1 / 2}
$$

By the Lemma $0 \in V(T)$. For any $\alpha \in \mathbf{C}, T-\alpha I$ is unbounded, so 0 $\in V(T-\alpha I)$. Hence $\alpha \in V(T)$.
The Lemma implies that, for $T$ defined on a subspace of $X$ with $V(T) \subset \mathbf{R}_{2}$ where we take $V(T)=\bigcup\{V(T, x): x \in S, T x$ defined $\}$, we have $\|T x\|^{2}$ $\leqslant M\|x\|\left\|T^{2} x\right\|$ with $M=8$. A result of Hille [3] implies that this holds with $M=2$, and an example of Kolmogorov [4] (differentiation on $L_{\infty}(\mathbf{R})$ ) shows that 2 is the best constant.

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