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ON A THEOREM OF BRICKMAN-FILLMORE

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ABSTRACT. Let V be a finite dimensional vector space over an arbitrary field. We show that if $\dim V \leqslant 3$ and if A, B and C are pairwise commuting linear transformations on V such that every subspace invariant for both A and B is also invariant for C, then C is a polynomial in A and B. (Brickman and Fillmore proved that if $B = 0$ then this statement is true for any finite dimensional vector space V.) An example shows that this is not true for $\dim V > 3$.

In [1] L. Brickman and P. A. Fillmore proved that if A and B are commuting linear transformations on a finite dimensional vector space over an arbitrary field and if every subspace invariant for A is also invariant for B, then B is a polynomial in A. Peter Fillmore suggested the following question (conveyed to me by Constantin Apostol):

If A, B and C are pairwise commuting linear transformations on a finite dimensional vector space V over an arbitrary field and if every subspace invariant for both A and B is also invariant for C, then is C a polynomial in A and B?

We shall prove that the answer to this question is true if the dimension of V is not more than 3 and false otherwise.

Suppose the dimension of V is 2. If A has no nontrivial invariant subspace then C is a polynomial in A by the Brickman-Fillmore result. If A is a scalar multiple of the identity then C is a polynomial in B. Similar statements can also be made for B. Finally, if A has a 1-dimensional eigenspace then A, B and C can be represented by upper triangular matrices relative to a fixed basis. By subtracting appropriate scalar multiples of the identity from A, B, and C, we may assume that:

$$A = \begin{pmatrix} 0 & a_1 \\ 0 & a_2 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & b_1 \\ 0 & b_2 \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} 0 & c_1 \\ 0 & c_2 \end{pmatrix}.$$

Since A and C commute we have $a_1 c_2 = c_1 a_2$. Thus (i) $a_1 \neq 0$ implies $(c_1/a_1)A = C$, (ii) $a_2 \neq 0$ implies $(c_2/a_2)A = C$ and (iii) $a_1 = a_2 = 0$ implies C is a polynomial in B.

The proof for the case when the dimension of V is 3 is obtained by considering the possible representations of A given by the rational decomposition theorem. We omit the details.

Finally, let

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$$A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

An easy computation shows that

$$AB = BA = AC = CA = BC = CB = 0 \quad \text{and} \quad A^2 = B^2 = C^2 = 0.$$

It follows from these that C is not a polynomial in A and B. To show that every subspace invariant under A and B is also invariant under C it is sufficient to consider cyclic subspaces (that is, subspaces generated by the action of A and B on a single vector). An easy calculation shows that if x is any vector, then Cx is a linear combination of Ax and Bx. This example can be extended to the case $\dim V > 4$ via direct sums.

Reference

1. L. Brickman and P. A. Fillmore, *The invariant subspace lattice of a linear transformation*, Canad. J. Math. **19** (1967), 810–822. MR **35** #4242.

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