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ON A FIXED POINT PROBLEM OF D. R. SMART

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In his book, *Fixed point theorems*, D. R. Smart poses the following problem which he says appears to be open: "Does every shrinking (i.e. contractive) mapping of the closed unit ball in a Banach space have a fixed point?" We answer this question in the negative by exhibiting a contractive mapping from the closed unit ball in a Banach space to itself which has no fixed point. Furthermore, our mapping has the additional properties that it is affine, a homeomorphism onto its image, and its inverse is Lipschitz.

Recall that C_0 is the Banach space of all real sequences $x = (x_1, x_2, \ldots)$ such that $\lim_{n\to\infty} x_n = 0$, and whose norm is defined by $\|x\| = \max\{|x_n|\}$.

We define our function as follows: Let $a_1, a_2, \ldots$ be any sequence of positive real numbers such that (i) each a_j is less than 1, and (ii) the sequence of partial products, $p_n = \prod_{j=1}^n a_j$, is bounded away from zero. (One such sequence is defined by $a_n = (2^n + 1)/(2^n + 2)$.) Now, if $x = (x_1, x_2, \ldots) \in C_0$, we let $f(x) = (1, a_1 x_1, a_2 x_2, a_3 x_3, \ldots)$. Then clearly if $\|x\| \leqq 1$, $\|f(x)\| \leqq 1$. (In fact, $\|f(x)\| = 1$, if $\|x\| \leqq 1$.) Thus f takes the unit ball in C_0 to itself. That f is affine (i.e. that $f(tx + (1-t)y) = tf(x) + (1-t)f(y)$) is trivial. Next, notice that

$$\|f(x) - f(y)\| = \max\{|a_n(x_n - y_n)|\} = a_j|x_j - y_j|,$$

for some j, and if $x \neq y$,

$$a_j|x_j - y_j| < |x_j - y_j| \leqq \max\{|x_n - y_n|\} = \|x - y\|.$$

Therefore, since $\|f(x) - f(y)\| < \|x - y\|$ if $x \neq y$, f is contractive.

Finally, suppose $x = (x_1, x_2, \ldots)$ is a fixed point of f. Then

$$x_1 = 1, \qquad x_3 = a_2 x_2 = a_1 a_2,$$
$$x_2 = a_1 x_1 = a_1, \qquad x_4 = a_3 x_3 = a_1 a_2 a_3, \text{ etc.}$$

and these numbers are bounded away from zero by the way that the sequence $a_1, a_2, \ldots$ was chosen. Thus, x is not in C_0 and the proof is complete.

REFERENCE

1. D. R. Smart, *Fixed point theorems*, Cambridge Univ. Press, New York, 1974.

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