PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 55, Number 1, February 1976

ON H-CLOSED SPACES

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ABSTRACT. A characterization of H-closed spaces in terms of projections is given along with relating properties.

Introduction. The primary purpose of this paper is to give a characterization of H-closed spaces which is an analogue to the following theorem for compact spaces: A space X is compact if and only if the projection from $X \times Y$ onto Y is a closed function for every space Y [9, p. 21].

Following the notation of [6], we utilize the notion of θ-closed subsets of a topological space from [11, p. 103] and our characterization is stated as follows:

THEOREM. *A Hausdorff space X is H-closed if and only if for every space Y, the projection from $X \times Y$ onto Y takes θ-closed subsets onto θ-closed subsets.*

Throughout, $\mathrm{cl}(K)$ will denote the closure of a set K.

Preliminary definitions and theorems.

DEFINITION 1. A net in a topological space is said to *θ-converge* (*θ-accumulate*) [6, Definition 3] to a point x in the space if the net is eventually (frequently) in $\mathrm{cl}(V)$ for each V open about x.

DEFINITION 2. A point x in a topological space X is in the *θ-closure* [11, p. 103] *of a set* $K \subset X$ (θ-$\mathrm{cl}(K)$) if $\mathrm{cl}(V) \cap K \neq \varnothing$ for any V open about x.

DEFINITION 3. A subset K of a topological space is *θ-closed* [11, p. 103] if it contains its θ-closure (i.e., θ-$\mathrm{cl}(K) \subset K$).

The following theorems give some parallels of properties of closure and closed sets in a topological space for θ-closure and θ-closed sets in the space and some relationships between these notions. The proofs of these theorems are straightforward and are omitted [11, Lemmas 1, 2, 3].

THEOREM 1. *A point x in a topological space is in the θ-closure of a subset K if and only if there is a net x_α in K which θ-converges to x* ($x_\alpha \xrightarrow[\theta]{} x$).

THEOREM 2. *In any topological space*

(a) *the empty set and the whole space are θ-closed*,

(b) *arbitrary intersections and finite unions of θ-closed sets are θ-closed*,

(c) $\mathrm{cl}(K) \subset \theta$-$\mathrm{cl}(K)$ *for each subset* K,

(d) *a θ-closed subset is closed.*

Received by the editors June 30, 1975.

AMS (MOS) subject classifications (1970). Primary 54D20.

Key words and phrases. H-closed spaces, θ-closed subsets, projections.

EXAMPLE 1. Each nonempty countable subset of the set of reals endowed with the cocountable topology is closed but not θ-closed.

Main results. There are several characterizations of H-closed spaces in the literature [2, p. 145], [1, p. 97]. For a definition, we use the following:

DEFINITION 4. A Hausdorff space X is H-closed if every open cover $\mathfrak{A}$ of X contains a finite subcollection $\mathfrak{B}$ such that $\{\mathrm{cl}(V): V \in \mathfrak{B}\}$ covers X.

We also make use of the following theorem immediately gotten from [11, Theorem 2]:

THEOREM 3. *A Hausdorff space is H-closed if and only if each net in the space has a θ-convergent subnet.*

DEFINITION 5. A function $g: X \to Y$ is *weakly continuous* [6, Theorem 6] if for each net x_α in X such that $x_\alpha \to x$, the net $g(x_\alpha) \underset{\theta}{\to} g(x)$.

DEFINITION 6. A function $g: X \to Y$ has a *strongly-closed graph* [6, p. 473] if for each $(x, y) \in (X \times Y) - G(g)$, there are open sets U and V about x and y, respectively, such that $(U \times \mathrm{cl}(V)) \cap G(g) = \varnothing$.

It is known that a function $g: X \to Y$ has a closed graph if and only if whenever a net $x_\alpha \to x$ in X and $g(x_\alpha) \to y$ in Y, it follows that $g(x) = y$ [13, p. 115]. We have the following similar result for functions with strongly-closed graphs.

THEOREM 4. *A function $g: X \to Y$ has a strongly-closed graph if and only if whenever a net $x_\alpha \to x$ in X and $g(x_\alpha) \underset{\theta}{\to} y$ in Y, it follows that $g(x) = y$.*

PROOF. Let g have a strongly-closed graph and let x_α be a net in X satisfying $x_\alpha \to x$ and $g(x_\alpha) \underset{\theta}{\to} y$. Then $(V \times \mathrm{cl}(W)) \cap G(g) \neq \varnothing$ for V, W open about x and y respectively. So, $(x, y) \in G(g)$ and $g(x) = y$. For the converse, let $(x, y) \in (X \times Y) - G(g)$. Then $y \neq g(x)$, and there is no net x_α in X satisfying $x_\alpha \to x$ and $g(x_\alpha) \underset{\theta}{\to} y$. If $(V_\sigma \times \mathrm{cl}(W_\xi)) \cap G(g) \neq \varnothing$ for each pair V_σ, W_ξ of sets open about x and y respectively, choose $(x_{\sigma,\xi}, g(x_{\sigma,\xi})) \in (V_\sigma \times \mathrm{cl}(W_\xi)) \cap G(g)$. The ordering of $\{V_\sigma \times \mathrm{cl}(W_\xi): V_\sigma, W_\xi$ open about x and y respectively$\}$ by inclusion renders $(x_{\sigma,\xi}, g(x_{\sigma,\xi}))$ a net with $x_{\sigma,\xi} \to x$ and $g(x_{\sigma,\xi}) \underset{\theta}{\to} y$, a contradiction. Therefore, there are sets V, W open about x, y, respectively, and satisfying $(V \times \mathrm{cl}(W)) \cap G(g) = \varnothing$; and $G(g)$ is strongly-closed. This completes the proof.

We may use the characterizations above to give a proof of the following theorem which is different and shorter than that given in [6]. If (x_α, D) is a net in a space X, we will denote $\{x_\alpha: \alpha > \mu\}$ by T_μ for each $\mu \in D$. Using this notation it is clear that x_α θ-converges (θ-accumulates) to a point $x \in X$ if for each open V about x, there is a $\mu \in D$ satisfying (each $\mu \in D$ satisfies) $T_\mu \subset \mathrm{cl}(V)$ ($T_\mu \cap \mathrm{cl}(V) \neq \varnothing$). Let $\mathfrak{S}$ denote a class of topological spaces containing the class of Hausdorff completely normal and fully normal spaces.

THEOREM 5. *A Hausdorff space Y is H-closed if and only if for every space in class $\mathfrak{S}$, each $g: X \to Y$ with a strongly-closed graph is weakly continuous.*

PROOF. Let Y be H-closed, let X be any space and let $g: X \to Y$ have a strongly-closed graph. Let $x_\alpha \to x$ in X. Then $g(x_\alpha)$ is a net in Y, so there is a subnet y_β of x_α and $y \in Y$ with $g(y_\beta) \underset{\theta}{\to} y$. By Theorem 4, $g(x) = y$. Let V be a regular open set about $g(x)$. If $g(x_\alpha)$ is not eventually in $\mathrm{cl}(V)$, there is a subnet z_μ of x_α such that $g(z_\mu)$ θ-converges to some $z \in Y - V$ since $Y - V$ is a regular closed set and thus H-closed. This then forces $g(x) \in Y - V$, a contradiction. So $g(x_\alpha) \underset{\theta}{\to} g(x)$. For the converse, let $x_0 \in Y$ and let (x_α, D) be a net in $Y - \{x_0\}$ with no θ-accumulation point in $Y - \{x_0\}$. Let $X = \{x_\alpha: \alpha \in D\} \cup \{x_0\}$ with the topology generated by $\{\{x_\alpha\}: \alpha \in D\}$ and $\{T_\mu \cup \{x_0\}: \mu \in D\}$ as the basic open sets. X is a Hausdorff door space [7, p. 76] and is easily shown to be in class $\mathcal{S}$. Let $i: X \to Y$ be the identity function and let $(x, y) \in (X \times Y) - G(i)$. If $x \neq x_0$, then $\{x\}$ is open in X; choose V open in Y about y with $x \notin \mathrm{cl}(V)$. Then, clearly, $(\{x\} \times \mathrm{cl}(V)) \cap G(i) = \varnothing$. If $x = x_0$, then $y \neq x_0$; so there is a $\mu \in D$ and a V open in Y about y satisfying $x_0 \notin \mathrm{cl}(V)$ and $T_\mu \cap \mathrm{cl}(V) = \varnothing$. So $X - \mathrm{cl}(V)$ is open in X about x and $[(X - \mathrm{cl}(V)) \times \mathrm{cl}(V)] \cap G(i) = \varnothing$. Thus, i has a strongly-closed graph and is weakly continuous. Consequently, if V is open about x_0, there is a $\mu \in D$ satisfying $T_\mu \subset \mathrm{cl}(V)$ [8, p. 44], so $x_\alpha \underset{\theta}{\to} x_0$. This completes the proof.

In [6, p. 474], an example is given to show that the strongly-closed graph condition in Theorem 5 cannot be relaxed to a closed graph condition. This example was extracted from [12] and is not described explicitly in [6] presumably because of its somewhat complicated description. We now exhibit a space with a simpler description which meets the purposes of the example in [6].

EXAMPLE 2. Let N be the set of positive integers and let $X = \{0\} \cup [1, \infty)$ with the topology generated by the usual subspace topology of the reals on $[1, \infty)$ and $\{\{0\} \cup \bigcup_{k=m}^{\infty}(k, k+1): m \in N\}$ as basic open sets.

(a) The space X is Hausdorff.

(b) The space X is not compact since N is an infinite subset of X without accumulation points.

(c) The space X is H-closed.

(d) The function g, from $\{1 + 1/n: n \in N\} \cup \{1\}$ with the subspace topology, defined by $g(1) = 1$ and $g(1 + 1/n) = n$ for each $n \in N$ has a closed graph which is not strongly-closed. Also, g is not weakly continuous at 1.

In [3], [4], [5], and [10], theorems of the following form are proved; X has property λ if and only if the projection $\pi_y: X \times Y \to Y$ is closed for each space Y in a certain class. The next four theorems and main results give an analogue of this form for H-closed spaces.

THEOREM 6. *If X is an H-closed space then the projection from $X \times Y$ onto Y takes θ-closed subsets onto θ-closed subsets for any space Y.*

PROOF. Let X be H-closed, let Y be any space and let $K \subset X \times Y$ be θ-closed. Let $y \in \theta\text{-}\mathrm{cl}(\pi_y(K))$. There is a net $(x_\alpha, y_\alpha) \in K$ with $y_\alpha \underset{\theta}{\to} y$. There is a subnet x_{α_μ} of x_α and $x \in X$ with $x_{\alpha_\mu} \underset{\theta}{\to} x$. So $(x_{\alpha_\mu}, y_{\alpha_\mu}) \underset{\theta}{\to} (x, y)$ and $(x, y) \in \theta\text{-}\mathrm{cl}(K) \subset K$. Thus $y \in \pi_y(K)$.

THEOREM 7. *If X is a Hausdorff space and the projection from $X \times Y$ onto Y*

takes θ-closed subsets onto closed subsets for every space Y, then X is H-closed.

PROOF. Let (y_α, D) be a net in X with no θ-convergent subnet and let $y_0 \notin X$. Let $Y = \{y_\alpha: \alpha \in D\} \cup \{y_0\}$ with the topology generated by $\{\{y_\alpha\}: \alpha \in D\}$ and $\{T_\mu \cup \{y_0\}: \mu \in D\}$ as basic open sets. Let $K = \{(y_\alpha, y_\alpha): \alpha \in D\}$ and let $(a, b) \in (X \times Y) - K$. Then $a \neq y_0$ and $a \neq b$. Let V be open about a satisfying $\{b, y_0\} \subset Y - \mathrm{cl}(V)$ and $T_\mu \subset Y - \mathrm{cl}(V)$ for some $\mu \in D$. Then $Y - \mathrm{cl}(V)$ is open and closed in Y and so $V \times (Y - \mathrm{cl}(V))$ is open about (a, b). Also,

$$\mathrm{cl}[V \times (Y - \mathrm{cl}(V))] \cap K = (\mathrm{cl}(V) \times (Y - \mathrm{cl}(V))) \cap K = \varnothing.$$

Thus, $(a, b) \notin \theta\text{-cl}(K)$ and K is θ-closed. $\pi_y(K)$ is therefore closed in Y and $y_0 \in \mathrm{cl}(\pi_y(K))$. This is a contradiction establishing the result.

Combining Theorems 6 and 7, we get the promised result.

THEOREM 8. *A Hausdorff space X is H-closed if and only if for every space Y, the projection from $X \times Y$ onto Y takes θ-closed subsets onto θ-closed subsets.*

Noting that the space Y used in the proof of Theorem 7 is a Hausdorff door space and is in the class $\mathfrak{S}$ whose description precedes Theorem 5, we may state the following theorem.

THEOREM 9. *A Hausdorff space X is H-closed if and only if for every Hausdorff door space (space in class $\mathfrak{S}$) Y, the projection from $X \times Y$ onto Y takes θ-closed subsets onto θ-closed subsets.*

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