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COMPUTATION OF THE UNORIENTED COBORDISM RING

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ABSTRACT. This note gives a set of generators for the unoriented cobordism ring and it gives a simplification of the algebra involved in Thom's computation of this ring.

In this note we give a simplification of the algebra involved in Thom's computation of the unoriented cobordism ring [3] and, along the way, a recursive formula for the generators of this ring given by Liulevicius [4].

Let A denote the mod two Steenrod algebra, A^* its dual and $\xi_i \in A^*$ the polynomial generators defined by Milnor [1]. In the following all homology and cohomology is with Z_2 coefficients. We identify $H_*(MO)$ with $H_*(BO)$ via the Thom isomorphism. The A module structure of $H^*(MO)$ defines an A^* comodule structure on $H_*(MO)$, ∇: $H_*(MO) \to A^* \otimes H_*(MO)$. The Whitney sum operation makes $H_*(MO)$ into an algebra and ∇ is an algebra map. Let T: $\prod^k BO_1 \to BO_k$ be the classifying map of the product of the canonical line bundles. T embeds $H^*(BO)$ in $H^*(\prod^\infty BO_1) = Z_2[t_1, t_2, \ldots]$ as the algebra of symmetric functions. For a partition $w = \{i_1, i_2, \ldots, i_k\}$ let s_w be the smallest symmetric function containing $t_1^{i_1} \cdots t_k^{i_k}$. Recall under the map $H^*(BO) \to H^*(BO) \otimes H^*(BO)$ induced by the Whitney sum,

$$s_w \to \sum s_{w_1} \otimes s_{w_2} \tag{1.1}$$

where the sum ranges over $w_1 \cup w_2 = w$. Let $\{x_w\} \subset H_*(BO)$ be the basis dual to $\{s_w\}$. Let $x_i = x_{\{i\}}$. (1.1) immediately yields: $x_w = x_{i_1} x_{i_2} \cdots x_{i_k}$ and hence $H_*(BO) = Z_2[x_1, x_2, \ldots]$. A straightforward calculation yields the following result of Switzer [2]: If $x = 1 + x_1 + x_2 + \ldots$ and $\xi = 1 + \xi_1 + \xi_2 + \ldots$,

$$\nabla x = \sum \xi^{i+1} \otimes x_i.$$

We define elements $y_i \in H_i(MO)$ by induction on i as follows: $y_0 = 1$, $y_i = x_i + \sum^{j<i} z_{ij} y_j$ where z_{ij} is the $i - j$ component of $(\sum y_{2^k - 1})^{j+1}$ if $j \neq 2^s - 1$ and $z_{ij} = 0$ if $j = 2^s - 1$.

THEOREM (1.2).

$$H_*(MO) = Z_2[y_1, y_2, \ldots], \qquad \nabla y_i = 1 \otimes y_i \quad \text{if } i \neq 2^j - 1,$$

$$\nabla y_{2^j - 1} = \sum \xi_{j-k}^{2^k} \otimes y_{2^k - 1}.$$

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The usual argument then gives

COROLLARY (1.3). *The Hurewicz map* $\rho: \pi_*(MO) \to H_*(MO)$ *defines an isomorphism* $\mathfrak{N}_* \approx Z_2[y_i | i \neq 2^j - 1]$ *where* $\mathfrak{N}_*$ *is the unoriented cobordism ring.*

REMARK. With analogous definitions for the y_i, (1.2) holds for $H_*(MU; Z_p)$, A_p', the algebra of reduced pth powers, and 2 replaced by p, an odd prime.

REMARK. Using the techniques in [5], one may show that $y_i = x_i$ if $i = tp^j - 2$, $t = 1, 2, \ldots, p-1$, $y_i \in H_i(MO; Z_2)$ for p even and $y_i \in H_{2i}(MU; Z_p)$ for p odd.

PROOF OF (1.2). Let $N = H_*(MO)/\{x_i | i = 2^k - 1, k > 0\}$ and let $p: H_*(MO) \to N$ be the projection. $A^* \otimes N$ is a polynomial algebra on the generators $\xi_i g$ and $p(x_i)$, $i \neq 2^k - 1$, and it is an A^* comodule under $\psi \otimes \mathrm{id}: A^* \otimes N \to A^* \otimes A^* \otimes N$ where ψ is the comultiplication in A^*. Let $f = (\mathrm{id} \otimes p)\nabla: H_*(MO) \to A^* \otimes N$. f is a ring homomorphism and an A^* comodule map. We show by induction on i that

$$
\begin{aligned}
f(y_i) &= 1 \otimes p(x_i), \qquad i \neq 2^k - 1, \\
&= \xi_k \otimes 1, \qquad i = 2^k - 1.
\end{aligned}
\tag{1.4}
$$

For $i = 0$ (1.4) is true. Suppose (1.4) is true for $1, 2, \ldots, i-1$.

$$f(y_i) = f(x_i) + f\left(\sum \left(\left(\sum_k y_{2^k-1}\right)^{j+1}\right)_{i-j} y_j\right)$$

where the above sum ranges over $j < i, j \neq 2^k - 1, k \geqslant 0$.

$$f(x_i) = \sum_{j=0}^{i} (\xi^{j+1})_{i-j} \otimes p(x_j).$$

Thus by the inductive hypothesis,

$$f(y_i) = 1 \otimes p(x_i) + (\xi)_i \otimes 1$$

and (1.4) holds for i.

f maps the y_i's onto polynomial generators and hence is an epimorphism. $H_*(MO)$ and $A^* \otimes N$ have the same rank in each dimension and therefore f is an isomorphism. (1.2) now follows from the comodule structure on $A^* \otimes N$.

BIBLIOGRAPHY

1. J. Milnor, *The Steenrod algebra and its dual*, Ann of Math. (2) **67** (1958), 150–171. MR **20** #6092.

2. R. Switzer, *Homology comodules*, Invent. Math. **20** (1973), 97–102.

3. R. Thom, *Quelques propriétés globales des variétés différentiables*, Comment. Math. Helv. **28** (1954), 17–86. MR **15**, 890.

4. A. Liulevicius, *A proof of Thom's theorem*, Comment. Math. Helv. **37** (1962/63), 121–131. MR **26** #3058.

5. E. Brown, D. Davis and F. Peterson, $H_*(BO)$, $H_*(BU)$ *and some results about the Steenrod algebra* (to appear).

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