PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 55, Number 1, February 1976

A COUNTEREXAMPLE CONCERNING INSEPARABLE FIELD EXTENSIONS

JAMES KEVIN DEVENEY

ABSTRACT. Let $K \supseteq M \supseteq k$ be a chain of fields of characteristic $p \neq 0$ where K is separable over M and M is purely inseparable over k. Recently it has been shown that if K has a separating transcendency basis over M or if M is of bounded exponent over k, then $K = M \otimes_k S$ where S is separable over k. This note presents an example to show that, in general, no such S need exist.

Throughout, we consider a chain of fields $K \supseteq M \supseteq k$ of characteristic $p \neq 0$ where K is separable over M and M is purely inseparable over k. Recent papers [1] and [2], have examined the question of when K can be expressed as $M \otimes_k S$ where S is a separable extension of k. It has been shown that if M is of bounded exponent over k [1, Theorem 5], or if K has a separating transcendency basis over M [2, Lemma 4], then $K = M \otimes_k S$ for some S. The purpose of this note is to provide an example to show that, in general, no such S exists. Necessarily, M will be of unbounded exponent over k and K will not have a separating transcendency basis over M.

EXAMPLE 1. Let P be a perfect field of characteristic $p \neq 0$ and let $\{x_1, x_2, \ldots, x_n, \cdots\}$ be an algebraically independent set over P. Set

$$K = P(x_1, \ldots, x_n, \cdots),$$
$$M = P(x_1x_2^p, x_2x_3^p, \ldots, x_nx_{n+1}^p, \cdots),$$
$$k = P(x_1^px_2^{p^2}, x_2^{p^2}x_3^{p^3}, \ldots, x_n^{p^n}x_{n+1}^{p^{n+1}}, \cdots).$$

Since $(x_nx_{n+1}^p)^{p^n} \in k$ for all n, M is purely inseparable over k. $\{x_1x_2^p, x_2x_3^p, \ldots, x_nx_{n+1}^p, \cdots\}$ is a p-basis for M and remains p-independent in K, so K is separable over M. Moreover, elementary calculations show $\{x_1x_2^p, \ldots, x_nx_{n+1}^p, \cdots\}$ is actually a p-basis for K, and thus K is relatively perfect over M, i.e. $K = M(K^p)$. We now assume there exists a field S separable over k such that $K = M \otimes_k S$.

LEMMA 2. *S is relatively perfect over k.*

PROOF. Recall that $K = M(K^p)$. Since we are assuming $K = M(S)$, $K^p = M^p(S^p)$, and so $K = M(M^p)(S^p) = M(S^p)$. Thus $K = M \otimes_k k(S^p)$ and we must have $S = k(S^p)$.

Now since S is relatively perfect over k, $S = k(S^{p^n}) \subseteq k(K^{p^n})$ for all n. Thus $S \subseteq \bigcap k(K^{p^n})$.

LEMMA 3. $\bigcap k(K^{p^n}) \subseteq P(x_1^p, x_2^{p^2}, \ldots, x_n^{p^n}, \cdots) = \bar{k}$.

PROOF. Since $\bar{k} \supseteq k$, $\bigcap k(K^{p^n}) \subseteq \bigcap \bar{k}(K^{p^n})$. Since

Received by the editors November 18, 1974.
AMS (MOS) subject classifications (1970). Primary 12F15.

$$K = \bar{k}(x_1) \otimes_{\bar{k}} \bar{k}(x_2) \otimes_{\bar{k}} \cdots \otimes_{\bar{k}} \bar{k}(x) \otimes_{\bar{k}} \cdots,$$

$\bigcap \bar{k}(K^{p^n}) = \bar{k}$, and the lemma is established.

We now have $S \subseteq \bigcap k(K^{p^n}) \subseteq \bar{k}$. To show no such S exists it suffices to show $M(\bar{k}) \neq K$.

$$\begin{aligned} M(\bar{k}) &= P(x_1 x_2^p, \ldots, x_n x_{n+1}^p, \cdots)\big(x_1^p, x_2^{p^2}, \ldots, x_n^{p^n}, \cdots\big) \\ &= P(x_1 x_2^p, \ldots, x_n x_{n+1}^p, \cdots)(x_1^p) = M(x_1^p). \end{aligned}$$

$P(x_1, \ldots, x_n)$ is algebraic over $P(x_1, x_1 x_2^p, \ldots, x_{n-1} x_n^p)$, and hence both fields have the same transcendence degree n over P, which means that x_1, $x_1 x_2^p, \ldots, x_{n-1} x_n^p$ are algebraically independent over P. Since this is true for all n, the set $\{x_1, x_1 x_2^p, x_2 x_3^p, \cdots\}$ is algebraically independent over P, and hence x_1 is transcendental over $M = P(x_1 x_2^p, x_2 x_3^p, \cdots)$. By Luroth's theorem $M(x_1^p) \subsetneq M(x_1) \subseteq K$. Thus no such S can exist.

References

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Department of Mathematical Sciences, Virginia Commonwealth University, Richmond, Virginia 23284