PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 55, Number 1, February 1976

SHORTER NOTES

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A SHORT PROOF OF THE UNIQUENESS OF HAAR MEASURE

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The purpose of this note is to give a brief proof of the uniqueness (up to a positive multiple) of left Haar measure μ (the existence of which we assume) on an arbitrary Hausdorff locally compact group G. The approach used here, employing the well-known device of an approximate identity, appears to be more transparent than any that we have found in the literature (e.g., [4, Theorem 29D, pp. 115–116] or [1, Theorem 1(B), pp. 15–16]). We remark that the elegant uniqueness proof for the Abelian case [5, 1.1.3, p. 2] cannot be improved upon; thus, our proof is of interest only for non-Abelian locally compact groups. We also observe that it is possible to give a combined existence and uniqueness proof (e.g., [2]).

We begin with some notation. Let ν denote a measure on G and let f, g be continuous functions on G with compact support. For such an f, let $f'(y) = f(y^{-1})$; also, for x in G, let $({}^x f)(y) = f(x^{-1}y)$. The convolution $f * g$ is defined as usual (using the Haar measure μ):

$$(f * g)(x) = \int_G f(y) g(y^{-1}x)\, d\mu(y) = \mu(f \cdot {}^x(g')) \tag{1}$$

and the convolution $\nu * f$ is the continuous function on G defined by

$$(\nu * f)(x) = \int_G f(y^{-1}x)\, d\nu(y) = \nu({}^x(f')). \tag{2}$$

Next, we recall two standard facts. First, there exists a (right) approximate identity (g_a) consisting of continuous functions on G with compact support; that is, (g_a) is a net with the property

$$\nu(f) = \lim_a \nu(f * g_a), \tag{3}$$

for every f and every ν. Second, by an application of Fubini's Theorem [3, Lemma A.2(iii), p. 179], it follows that

$$\nu(f * g) = \mu(f \cdot (\nu * g')), \tag{4}$$

for every f, g and every ν.

Received by the editors July 15, 1975.

AMS (MOS) subject classifications (1970). Primary 22D05, 28A70, 43A05.

Key words and phrases. Locally compact group, Haar measure, translation invariant measure, approximate identity.

THEOREM. *If ν is a left translation invariant measure on G, then ν is a complex multiple of μ.*

PROOF. Let (g_a) be an approximate identity; then for every f, (3) obtains. Rewriting (3), using (4), yields

$$\nu(f) = \lim_a \mu(f \cdot (\nu * g_a')), \tag{5}$$

for every f. However, since ν is left translation invariant, we have

$$(\nu * g_a')(x) = \nu({}^x(g_a)) = \nu(g_a), \tag{6}$$

for each x in G and each a. Consequently, (5) becomes

$$\nu(f) = \lim_a \mu(f \cdot \nu(g_a)) = \lim_a \nu(g_a) \cdot \mu(f) = \left(\lim_a \nu(g_a)\right) \cdot \mu(f), \tag{7}$$

for every f. Finally, by choosing f so that $\mu(f)$ is nonzero, it follows that $\lim_a \nu(g_a)$ is equal to a constant c and that $\nu = c \cdot \mu$. Q.E.D.

ADDED IN PROOF. We should observe that the existence of a net (g_a) satisfying (3) does not depend upon the essential uniqueness of Haar measure; in particular, the modular function is not involved. Indeed, if for every compact symmetric neighborhood a of the identity we let g_a be a symmetric ($g_a = g_a'$) positive continuous function on G supported in a with $\mu(g_a) = 1$, then a straightforward argument using the uniform continuity of f and the left translation invariance of μ yields (3).

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