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NUMERICAL RANGE OF A WEIGHTED SHIFT WITH PERIODIC WEIGHTS

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ABSTRACT. Calculation of the numerical range of a weighted shift is reduced to the solution of a polynomial equation when the weights form a periodic sequence, or approach a periodic sequence from below.

Introduction. A *weighted shift* on l^2 or l^2_+ is a linear operator S defined by $Se_n = s_n e_{n+1}$ where $\{e_n\}$ is an orthonormal basis, and $\{s_n\}$ a sequence of complex numbers. The *numerical range* of an operator S is the set of complex numbers (Sx, x) where $\|x\| = 1$; this is denoted $W(S)$. For definiteness we assume here a one-sided shift, indexed by positive integers; the proofs and results are the same for a two-sided shift.

Then we are given

$$x = x_1 e_1 + x_2 e_2 + \cdots, \quad x_i \text{ complex}, \qquad \sum |x_i|^2 < \infty;$$

$$Sx = x_1 s_1 e_2 + x_2 s_2 e_3 + \cdots.$$

We begin with some simple facts about weighted shifts [1].

(1) S is a bounded operator if and only if $\{s_n\}$ is a bounded sequence, and then $\|S\| = \sup|s_n|$.

(2) S is unitarily equivalent to a shift with weights t_i whenever $|t_i| = |s_i|$ for all i. In particular, S is unitarily equivalent to cS whenever $|c| = 1$.

(3) Therefore $W(S)$ has circular symmetry about 0: $cW = W$ whenever $|c| = 1$.

(4) Since W is convex, it follows that $W(S)$ is a disk centered at 0; its radius $w(S)$ is the *numerical radius* of S.

It is an easy exercise to find $W(S)$ in some special cases. For example:

(5) If $|s_n| \leqslant K$ for all n, and $|s_n| \to K$, then $W(S) = K$.

By (2) it suffices to consider shifts with real nonnegative weights, $s_n \geqslant 0$, and we shall do so.

THEOREM 1. *If $\{s_n\}$ is a periodic sequence, of period r, then*

$$w(S) = \max\{s_1 x_1 x_2 + s_2 x_2 x_3 + \cdots + s_r x_r x_1 : \quad x_i \text{ real}, \quad x_1^2 + \cdots + x_r^2 = 1\},$$

and finding this is equivalent to solving a polynomial equation of degree r.

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PROOF. First consider a sequence x consisting of the finite sequence of complex numbers $\{x_1, x_2, \ldots, x_r\}$ repeated k times, with 0's thereafter. Then

$$\begin{aligned} Sx &= \{0, [s_1 x_1, s_2 x_2, \ldots, s_r x_r], \text{(repeated } k \text{ times)}, 0, 0, \ldots\}, \\ (Sx, x) &= k(s_1 x_1 x_2 + s_2 x_2 x_3 + \cdots + s_r x_r x_1) - s_r x_r x_1, \\ (x, x) &= k(|x_1|^2 + |x_2|^2 + \cdots + |x_r|^2), \end{aligned}$$

and for large k we see that $(Sx, x)/(x, x)$ can be made arbitrarily close to

$$\frac{s_1 x_1 x_2 + s_2 x_2 x_3 + \cdots + s_r x_r x_1}{|x_1|^2 + |x_2|^2 + \cdots + |x_r|^2}. \tag{1}$$

Therefore $w(S)$ is at least equal to

$$\max\{|s_1 x_1 x_2 + \cdots + s_r x_r x_1|: x_i \text{ complex}, \ |x_1|^2 + \cdots + |x_r|^2 = 1\}.$$

By multiplying x_k by $e^{i\theta_k}$ we may make these components real and nonnegative: this gives the problem:

$$\begin{gathered} \text{Maximize } s_1 x_1 x_2 + s_2 x_2 x_3 + \cdots + s_r x_r x_1 \\ \text{subject to } x_1^2 + \cdots + x_r^2 = 1, \ s_k, x_k \text{ real}. \end{gathered}$$

The use of Lagrange multipliers gives the system:

$$\begin{aligned} s_r x_r + s_1 x_2 &= \lambda x_1 \\ s_1 x_1 + s_2 x_3 &= \lambda x_2 \\ &\cdots \\ s_{r-1} x_{r-1} + s_r x_1 &= \lambda x_r. \end{aligned}$$

Elimination of the x_i gives a polynomial equation in λ of degree r; $x_2, \ldots, x_r$ are found by substitution (in terms of x_1), and x_1 is then found by the relation $x_1^2 + \cdots + x_r^2 = 1$.

We now establish that $w(S)$ is no greater than this maximum value of $s_1 x_1 x_2 + \cdots + s_r x_r x_1$.

LEMMA. *If a_k, b_k are nonnegative constants with $b_k \neq 0$, then*

$$\frac{a_1 + a_2 + \cdots}{b_1 + b_2 + \cdots} \leqslant \sup_k \frac{a_k}{b_k}$$

whenever the left side is defined.

PROOF. We first show this for finite sums. If $a/c \geqslant b/d$, then

$$\frac{a+b}{c+d} \leqslant \frac{a + ad/c}{c+d} = \frac{a}{c}$$

and the result for finite sums follows by induction. Then

$$\frac{a_1 + a_2 + \cdots + a_n}{b_1 + b_2 + \cdots + b_n} \leqslant \max_{k \leqslant n} \frac{a_k}{b_k} \leqslant \sup_{k \geqslant 1} \frac{a_k}{b_k}$$

and hence the lim sup of the left side satisfies the same inequality; this proves the lemma.

Resuming the proof of Theorem 1: suppose $|x| = 1$, and write the components of x as

$$x = \{a_{11}a_{12}\cdots a_{1r};a_{21}a_{22}\cdots a_{2r};\dots\},$$

a_{ij} complex. Then

$$\frac{|(Sx,x)|}{(x,x)} = \frac{|s_1a_{11}a_{12}+\cdots+s_ra_{1r}a_{21}+s_1a_{21}a_{22}+\cdots+s_ra_{2r}a_{31}+\cdots|}{|a_{11}|^2+\cdots+|a_{1r}|^2+|a_{21}|^2+\cdots+|a_{2r}|^2+\cdots}$$

$$\text{(1)}\qquad \leqslant \sup_k \frac{|s_1a_{k1}a_{k2}+\cdots+s_ra_{kr}a_{(k+1)1}|}{|a_{k1}|^2+\cdots+|a_{kr}|^2}$$

and

$$\text{(2)}\qquad \leqslant \sup_k \frac{|s_1a_{k1}a_{k2}+\cdots+s_ra_{kr}a_{(k+1)r}|}{|a_{(k+1)1}|^2+|a_{k2}|^2+\cdots+|a_{kr}|^2}.$$

Inequality (1) follows from the lemma, and (2) follows by deleting $|a_{11}|^2$ from the denominator (thus increasing the value of the fraction), regrouping terms of the denominator, and applying the lemma.

Setting

$$x_1 = \max(|a_{k1}|,|a_{(k+1)1}|),\quad x_j = |a_{kj}|\quad \text{for } j = 2,\dots,r,$$

we see that

$$\frac{|(Sx,x)|}{(x,x)} \leqslant \max\{s_1x_1x_2+\cdots+s_rx_rx_1:\ x_1^2+\cdots+x_r^2 = 1, x_i \text{ real}\},$$

and therefore $w(S)$ is equal to this maximum value; this completes the proof of Theorem 1.

THEOREM 2. *If $s_k \leqslant p_k$ and $s_k - p_k \to 0$ as $k \to \infty$, where $\{p_k\}$ is a periodic sequence with period r, then*

$$w(S) = \max\{p_1x_1x_2+\cdots+p_rx_rx_1:\ x_1^2+\cdots+x_r^2 = 1, x_i \textit{ real}\}.$$

PROOF. Given $\varepsilon > 0$, letting T be the shift with weights p_k, let x be a unit vector such that $(Tx,x) > w(T) - \varepsilon$. Choose n such that $|s_k - p_k| < \varepsilon$ for $k \geqslant n$. Let y be the unit vector with $y_k = 0$, $k = 1,\dots,n$; $y_{k+n} = x_k$, $k = 1, 2, \dots$. Then $(Ty,y) = (Tx,x) > w(T) - \varepsilon$.

Now

$$\|Ty - Sy\| \leqslant \sup_{k>n}|p_k - s_k| \leqslant \varepsilon$$

so $|(Ty,y) - (Sy,y)| < \varepsilon$ and so $(Sy,y) > w(T) - 2\varepsilon$.

Therefore $w(S) \geqslant w(T)$. Since $0 \leqslant s_k \leqslant p_k$, we easily have $w(S) \leqslant w(T)$, and so the two are equal. By Theorem 1,

$$w(T) = \max\{p_1x_1x_2+\cdots+p_rx_rx_1:\ x_1^2+\cdots+x_r^2 = 1, x_i \text{ real}\} = w(S);$$

this proves Theorem 2.

EXAMPLES. (1) If $r = 2$ the weights are $a, b, a, b, \ldots$; we are to maximize $(a+b)x_1x_2$, that is, maximize x_1x_2 subject to $x_1^2 + x_2^2 = 1$. The solution is $x_1 = x_2 = 1/\sqrt{2}$, $(a+b)x_1x_2 = (a+b)/2$, and so the numerical radius is the average of the two weights.

(2) If $r = 3$ the weights are $a, b, c, a, b, c, \ldots$; we must maximize $ax_1x_2 + bx_2x_3 + cx_3x_1$ with $x_1^2 + \cdots + x_3^2 = 1$; we have the system

$$ax_2 + cx_3 = \lambda x_1, \quad ax_1 + bx_3 = \lambda x_2, \quad bx_2 + cx_1 = \lambda x_3,$$

which (if $x_1 \neq 0$) leads to the cubic equation $\lambda^3 - (a^2 + b^2 + c^2)\lambda - 2abc = 0$.

NOTES. (1) The numerical range is always a disk about 0, of positive radius except in the trivial case where all the weights are zero.

(2) If any weight is zero then the disk is closed; for (assuming $s_r = 0$ for example) then $(Sx, x)/(x, x)$ is actually equal to the expression (1), which in turn attains its maximum on the compact sphere (2).

(3) For weights $(1, 1, 1, \ldots)$ the disk is open; for $|(Sx, x)| = 1$, $|x| = 1$, would imply $Sx = kx$, $|k| = 1$, which is impossible.

(4) I surmise, but have yet to prove, that the disk is open whenever all the weights are nonzero; that is, (Sx, x) cannot attain its sup $w(S)$ for $|x| = 1$.

REFERENCE

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