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CONJUGACY IN ABELIAN-BY-CYCLIC GROUPS

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ABSTRACT. It is shown that each finitely generated torsion-free abelian-by-cyclic group has solvable conjugacy problem. This is done by showing that solving the conjugacy problem for these groups is equivalent to a certain decision problem for modules over the complex group algebra of an infinite cyclic group.

1.1 **Introduction.** Let G be an abelian-by-cyclic group. That is, G has an abelian normal subgroup A with $G/A = T$ cyclic. If G is finitely generated, it can be recursively presented, so it is meaningful to ask whether G has solvable conjugacy problem. Our aim is to answer this question affirmatively when G is torsion-free. More precisely, we prove

THEOREM 1. *Let A be an abelian normal subgroup of the finitely generated group G. If $G/A = T$ is cyclic and A has no elements of finite order, then G has solvable conjugacy problem.*

1.2 **Preliminaries.** We use the standard notation

$$x^y = y^{-1}xy, \qquad [x,y] = x^{-1}y^{-1}xy.$$

If R is a commutative ring with identity and T is a group, RT is the R-algebra which is additively the free R-module with basis T, with multiplication induced by the multiplication in T. In particular, ZT is the *integral group ring of T* and $\mathbf{C}T$ is the *complex group algebra of T*. If T is infinite cyclic, $\mathbf{C}T$ is a principal ideal domain.

Let A be an abelian normal subgroup of a group G and let $T = G/A$. Conjugation in G induces an action of T on A which gives A the structure of a ZT-module. If we write A additively, then $at = g^{-1}ag$ where $g \in G$ is such that $gA = t$.

If G is finitely generated and T is finitely presented, A is finitely generated as a ZT-module.

If T is infinite cyclic, the short exact sequence $0 \to A \to G \to T \to 1$ splits, so that G is isomorphic with the semidirect product $A]T$ of the ZT-module A by T. Thus T may be regarded as a subgroup of G.

If A is a ZT-module, $A \otimes_Z \mathbf{C}$ becomes a $\mathbf{C}T$-module via $(a \otimes 1)t = at \otimes 1$. If A is generated as a ZT-module by $a_1, \ldots, a_k$, $A \otimes \mathbf{C}$ is generated as a $\mathbf{C}T$-module by $a_1 \otimes 1, \ldots, a_k \otimes 1$. If A is torsion-free, the map $a \mapsto a \otimes 1$ defines an embedding (of abelian groups) of A into $A \otimes \mathbf{C}$.

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1.3 **A module-theoretic decision problem.** Let R be a commutative ring with identity, T a group and A a finitely presented RT-module. We say A has *solvable conjugacy problem* if there is an effective procedure for determining whether, for a and b in A, there is $t \in T$ with $at = b$.

THEOREM 2. *Let T be cyclic and A a finitely generated ZT-module without elements of finite order. Then A has solvable conjugacy problem.*

To prove Theorem 2, notice first that if T is a finite cyclic group, A is finitely generated as an abelian group and so $A]T$ is polycyclic. By [2] $A]T$ has solvable conjugacy problem, so A has solvable conjugacy problem. Thus we may assume T is infinite cyclic.

Now, since A is torsion-free, the map $a \mapsto a \otimes 1$ induces an embedding of A into $A \otimes \mathbf{C}$, and it is enough to show that the $\mathbf{C}T$-module $A \otimes \mathbf{C}$ has solvable conjugacy problem.

Since $\mathbf{C}T$ is a principal ideal domain, there is a decomposition

$$A \otimes \mathbf{C} = A_1 \oplus \cdots \oplus A_k$$

where each A_i $(1 \leqq i \leqq k)$ is a nonzero cyclic submodule of $A \otimes \mathbf{C}$. Clearly, then, it is enough to show that a cyclic $\mathbf{C}T$-module has solvable conjugacy problem.

Let $A_i \approx \mathbf{C}T/J_i$ $(1 \leqq i \leqq k)$. We distinguish between the cases $J_i = (0)$ and $J_i \neq (0)$.

1.4 **Free submodules.** If $J_i = (0)$, $A_i \approx \mathbf{C}T$ is a free submodule of rank 1.

To see that $\mathbf{C}T$ has solvable conjugacy problem, let t be a generator for T. Then there are unique representations

$$a = c_1 t^{s_1} + \cdots + c_p t^{s_p}, \quad b = d_1 t^{w_1} + \cdots + d_q t^{w_q}$$

with

$$0 \neq c_i \in \mathbf{C} \quad (1 \leqq i \leqq p) \quad \text{and} \quad s_1 < \cdots < s_p,$$

$$0 \neq d_i \in \mathbf{C} \quad (1 \leqq i \leqq q), \quad w_1 < \cdots < w_q.$$

To decide whether $at^l = b$ for some $l \in Z$, simply notice that the only possibility for l is $w_q - s_p$. It is clearly possible, for a fixed power t^l of t, to effectively decide whether $at^l = b$.

1.5 **Finite-dimensional submodules.** If $J_i \neq (0)$, $\dim_{\mathbf{C}} A_i = n < \infty$. The action of a generator t of T on a basis of the vector space A_i yields an invertible $n \times n$ matrix M in the usual way. Notice that the matrix M is effectively computable from a presentation of A_i.

Recall that if G is a finitely presented group and the word problem is solvable for some finite presentation P_1 of G, then it is also solvable for any other finite presentation P_2 of G (cf. [4]). A similar argument shows that if the conjugacy problem is solvable for some finite presentation P_1 of a $\mathbf{C}T$-module A, it is solvable for any other presentation P_2 of A. Because of this we may assume that the module A_i is presented so that the matrix M is in Jordan canonical form (cf. [3]). Thus

$$M = \begin{pmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_q \end{pmatrix}$$

where each J_i $(1 \leqq i \leqq q)$ is of the form

$$J_i = \begin{pmatrix} \lambda & 1 & \cdot & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & 1 \\ 0 & \cdot & \cdot & \cdots & \lambda \end{pmatrix}.$$

Let

$$a = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} b_1 \\ \vdots \\ b_n, \end{pmatrix}, \qquad a_i, b_i \in C \quad (1 \leqslant i \leqslant n).$$

Our aim is to show there is an effective process for determining whether $M^r a = b$ for some positive integer r. Since

$$M = \begin{pmatrix} J_1^r & & & \\ & J_2^r & & \\ & & \ddots & \\ & & & J_q^r \end{pmatrix}$$

it is not hard to see that we may assume $k = 1$ so that M can be taken to be the $n \times n$ matrix

$$M = \begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & 1 \\ 0 & \cdot & \cdot & \cdots & \lambda \end{pmatrix} = \lambda I + S$$

where

$$S = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & 1 \\ 0 & \cdot & \cdot & \cdots & 0 \end{pmatrix}.$$

Here, $\lambda \neq 0$ since M is invertible.

$$M^r = (\lambda I + S)^r = \sum_{k=0}^{r} \binom{r}{k} \lambda^{r-k} S^k.$$

Since $S^n = 0$, we have for $r \geqq n$,

$$n^r = \sum_{k=0}^{n-1} \binom{r}{k} \lambda^{r-k} S^k.$$

To effectively decide whether $M^r a = b$ for some r, then, we must be able to decide in a finite number of steps whether there is a solution to the system of equations:

$$\begin{array}{llllllll}
(1) & \lambda^r a_1 + & r\lambda^{r-1} a_2 & + \cdots & \cdots & + \binom{r}{n-1} \lambda^{r-n+1} a_n & = & b_1, \\
(2) & \cdot & \lambda^r a_2 & + \cdots & \cdots & + \binom{r}{n-2} \lambda^{r-n+2} a_n & = & b_2, \\
\vdots & & \vdots & & & \vdots & & \\
(n-1) & & & & \lambda^r a_{n-1} + & r\lambda^{r-1} a_n & = & b_{n-1}, \\
(n) & & & & & \lambda^r a_n & = & b_n.
\end{array}$$

We can obtain a solution as follows. Suppose first that $|\lambda| \neq 1$, and consider equation (n): $\lambda^r a_n = b_n$.

As r increases without bound, $|\lambda^r|$ either increases without bound or approaches 0. Either way, if $a_n \neq 0$ there are only finitely many values of r which we need check and we can check each of these values simply by computing $M^r a$. If $a_n = 0$, consider equation $(n-1)$, which becomes $\lambda^r a_{n-1} = b_{n-1}$.

Again, considering the cases $a_{n-1} \neq 0$ and $a_{n-1} = 0$, we can inductively obtain a decision in a finite number of steps.

Now assume $|\lambda| = 1$. By an induction argument, we may assume $a_n \neq 0$. Consider equation $(n-1)$: $\lambda^r a_{n-1} + r\lambda^{r-1} a_n = b_{n-1}$. Since $a_n \neq 0$ and $|\lambda| = 1$, $|\lambda^r a_{n-1} + r\lambda^{r-1} a_n|$ increases without bound as r increases without bound. Thus, there are again only finitely many values of r for which we need compute $M^r a$.

We have shown that we can effectively decide whether $M^r a = b$ for $r \geqq 1$. By considering $(M^{-1})^r = M^{-r}$, we can decide in a finite number of steps whether $M^r a = b$ for any $r \in Z$. This completes the proof of Theorem 2.

1.6 **Proof of Theorem 1.** Let G, A and T be as in the statement of Theorem 1. We have seen that we may assume T is infinite cyclic, so that $G = A]T$. Let $g_1, g_2 \in G$. If $g_1 \neq g_2 \bmod A$, then g_1 and g_2 are not conjugate in G. Thus there are unique representations

$$g_1 = as \quad \text{and} \quad g_2 = bs \qquad (a, b \in A, s \in T).$$

To decide whether g_1 is conjugate to g_2 by an element $g_3 = ct$ $(c \in A, t \in T)$, we must check whether $g_1^{g_3} = (as)^{ct} = b$. Now

$$(as)^{ct} = a^t (c^{-1} sc)^t = (a^t c^{-t} c^{s^{-1} t}) s.$$

If we write A additively, we have g_1 conjugate to g_2 if and only if there is a solution $c \in C$, $t \in T$ to the equation $at + (s^{-1} - 1)t = b$.

When we look at this equation modulo the normal subgroup N of G generated by s (notice that $N \cap A$ is the submodule of A generated by $(s^{-1} - 1)$), we see that there is a solution if and only if $at = b \bmod N$.

If $N \neq 1$, then G/N is polycyclic and so has solvable conjugacy problem. If $N = 1$, then $s = 1$ and invoking Theorem 2 proves that G has solvable conjugacy problem. This completes the proof of Theorem 1.

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REFERENCES

1. J. Boler, *Embedding and conjugacy in metabelian groups*, Thesis, Rice University, 1974.

2. E. Formanek, *Matrix techniques in polycyclic groups*, Thesis, Rice University, 1970.

3. I. N. Herstein, *Topics in algebra*, Blaisdell, New York and Toronto, 1964, p. 258. MR **30** #2028.

4. W. Magnus, A. Karass and D. Solitar, *Combinatorial group theory*: *Presentations of groups in terms of generators and relations*, Pure and Appl. Math., vol. 8, Interscience, New York, 1966, p. 55. MR **34** #7617.

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