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PROOF OF A CONJECTURE OF FRIEDMAN

DONALD A. MARTIN

ABSTRACT. We prove that every uncountable hyperarithmetic set has a member of each hyperdegree $\geqslant \mathbf{0}$, the hyperdegree of Kleene's 0.

We improve the main result of Friedman [1] by proving his conjecture that every uncountable hyperarithmetic set has a member of each hyperdegree $\geqslant \mathbf{0}$, the hyperdegree of Kleene's 0. This result has been obtained independently by Friedman by a different method. Friedman's proof uses ideas employed by L. Harrington to obtain a partial result.

In [2] it is shown that there is a function f: $\omega \to \omega$ of hyperdegree $\mathbf{0}$ and a Gödel number e such that $(\forall k)(f(k) \leqslant g(k))$ implies that f is hyperarithmetic in g with Gödel number e.

By [1] it suffices to prove that each recursive tree of finite sequences of natural numbers with uncountably many branches has a branch of each hyperdegree $\geqslant \mathbf{0}$. Let T be such a tree and let f be as above. Let x: $\omega \to 2$ have hyperdegree $\geqslant \mathbf{0}$. Let A be the Cantor-Bendixson perfect subtree of T. $A \in \Sigma_1^1$. A point in A is *good* if it has at least two immediate successors in A.

We shall define a branch h through A. Let $\sigma_0 < \sigma_1 < \cdots$ be the good points on our branch. $\ln(\sigma)$ is the length of the finite sequence σ. $h(\ln(\sigma_n))$ will always be given either the smallest or the next to smallest possible value. $h(\ln(\sigma_{2n}))$ and $h(\ln(\sigma_{2n+1}))$ will have the minimal possible value unless there are exactly k numbers $m < n$ such that

$$h(\ln(\sigma_{2m})) \text{ is not minimal, and } f(k) \leqslant \ln(\sigma_{2n}).$$

When this happens, $h(\ln(\sigma_{2n}))$ will not be minimal and $h(\ln(\sigma_{2n+1}))$ will be minimal just in case $x(k) = 0$.

For g: $\omega \to \omega$ let $g \in C$ if and only if, for each k, there are at least $2k + 1$ good points σ_n of length $\leqslant g(k)$ such that $h(\ln(\sigma_n))$ is not minimal. C is Σ_1^1 in h and $g \in C \Rightarrow (\forall k)(g(k) \geqslant f(k))$. f is hyperarithmetic in h since

$$f^* = f \Leftrightarrow (\exists g \in C)\ (f^* \text{ is hyperarithmetic in } g \text{ with Gödel number } e).$$

Since both f and A are hyperarithmetic in each of x and h, x and h are hyperarithmetic in one another.

REFERENCES

1. H. Friedman, *Borel sets and hyperdegrees*, J. Symbolic Logic **38** (1973), 405–409.
2. C. Jockusch and R. Soare, *Encodability of Kleene's* 0, J. Symbolic Logic **38** (1973), 437–440.

THE LOGIC GROUP, ROCKERFELLER UNIVERSITY, NEW YORK, NEW YORK 10021

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