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ON THE EMBEDDING OF SCHWARTZ SPACES INTO PRODUCT SPACES

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ABSTRACT. Every Schwartz space is embeddable into some sufficiently high power E^I of a given Banach space E if and only if E contains l_n^∞ uniformly.

In [13] Saxon showed that every nuclear space can be embedded in some sufficiently high power of every Banach space. In [2] Diestel and Lohman showed that a locally convex space that is embeddable in some sufficiently high power of every Banach space is a Schwartz space. In [11] we showed that every Schwartz space can be embedded in some sufficiently high power of c_0. In [1] Bellenot gave examples of Schwartz spaces that are not embeddable in any power of l_p ($1 < p < \infty$). In Theorem 2 below we characterize those Banach spaces E such that every Schwartz space is embeddable in some sufficiently high power of E. In particular, Theorem 2 implies that every Schwartz space is embeddable in some sufficiently high power of every $\mathcal{L}_\infty$-space, and Corollary 5 implies that if $\{E_\nu\}$ is a family of Banach spaces such that each E_ν is an $\mathcal{L}_{p_\nu}$-space ($1 \leqslant p_\nu < \infty$), then there is a Schwartz space that is not embeddable in any power of $\prod E_\nu$.

In [12] (and independently in [6]) it was shown that c_0, equipped with the topology of uniform convergence on null sequences in the norm dual of c_0 is a universal Schwartz space; and in [12] we asked whether every $\mathcal{L}_\infty$-space, equipped with the topology of uniform convergence on null sequences in the norm dual, is a universal Schwartz space. Theorem 2 below gives an affirmative answer to this question.

The proof of our main result (Theorem 2) depends in a very essential way on the important results of Figiel [3] concerning the factorization of compact linear operators through Banach spaces. Parts of the proof of Theorem 2 are similar to some of the proofs appearing in [11] and [12].

A linear operator between locally convex spaces is compact if it transforms some neighborhood of 0 into a relatively compact set. A locally convex space E is a Schwartz space if every continuous linear operator from E into a Banach space is compact. If T: $E \to F$ is a compact linear operator between Banach spaces, then (by [15, Theorem 1, p. 76]) there is a null sequence $\{a_n\}$ in the norm dual E' of E such that $\|Tx\| \leqslant \sup|\langle x,a_n\rangle|$ for every x in E; consequently, if E and F are Banach spaces and $\mathfrak{S}$ denotes the topology on E

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of uniform convergence on null sequences in E', then (i) a linear operator T: $E \to F$ is compact if and only if T: $E[\mathfrak{S}] \to F$ is continuous, and (ii) $E[\mathfrak{S}]$ is a Schwartz space. A Schwartz space E is universal if every Schwartz space is linearly homeomorphic to a linear subspace of some sufficiently high power E^I of E. A projective limit proj lim $f_{\mu\nu}(E_\nu)$ of Banach spaces $\{E_\nu\}$ (for the definition of projective limit see [14, p. 52]) is compact if for each μ there is a $\nu > \mu$ such that $f_{\mu\nu}$: $E_\nu \to E_\mu$ is compact. If each $E_\nu = E$, we say that proj lim $f_{\mu\nu}(E_\nu)$ is a projective limit of E-spaces and write proj lim $f_{\mu\nu}(E)$.

A continuous linear operator T: $E \to F$ between locally convex spaces factors through a subspace of a locally convex space G if there exist continuous linear operators P: $E \to G$ and Q: $P(E) \to F$ such that $T = QP$. A locally convex space E has the subspace factorization property if every compact linear operator between Banach spaces factors through a subspace of E. For a normed space E let $c_n(E) = \inf\{d(H, l_n^\infty)$: H is an n-dimensional linear subspace of $E\}$, where $d(H, l_n^\infty)$ denotes the Banach-Mazur distance between H and l_n^∞ and where l_n^∞ denotes the vector space of all n-tuples $\lambda = (\lambda_1, \dots, \lambda_n)$ of scalars equipped with the norm $\|\lambda\|_\infty = \sup|\lambda_k|$. A normed space E contains l_n^∞ uniformly if $c_n(E) = 1$ for all n. Figiel [3, Theorem 6.1, p. 202] has shown that a Banach space has the subspace factorization property if and only if it contains l_n^∞ uniformly. By combining Proposition 73 and Théorèmes 92 and 93 of Maurey [10] it follows that for $1 < p \leqslant 2$ a linear subspace of a quotient space of an $L^p(X, \nu)$ measure space never contains l_n^∞ uniformly. The following proposition implies that a $\mathfrak{L}_p$-space (for the definition and basic properties of $\mathfrak{L}_p$-spaces see [7], [8] and [9]) contains l_n^∞ uniformly if and only if $p = \infty$. The proof of the following proposition is based on a modification of the proof of [4, Theorem 9, p. 359]. The author would like to thank both T. Figiel and D. J. H. Garling for their comments concerning the constants $c_n(E)$ as well as their references to the work of Maurey [10].

PROPOSITION 1. *Let E be a real $\mathfrak{L}_{p,\lambda}$-space and let K_G denote the universal Grothendieck constant* [7, *p.* 279].

(a) *If $p = \infty$, then $c_n(E) = 1$.*

(b) *If $2 \leqslant p < \infty$, then $c_n(E) \geqslant n^{1/p}/(\lambda K_G)$.*

(c) *If $1 \leqslant p \leqslant 2$, then $c_n(E) \geqslant n^{1/2}/(\lambda K_G)$.*

[We note that if E is complex, then K_G in (b) and (c) above can be replaced by $2K_G$ (see the remark at the top of [7, p. 282]).]

PROOF. For any unexplained notation or terminology see [4] and [7]. If E is an $\mathfrak{L}_\infty$-space, then the sequence $\{c_n(E)\}$ is bounded and, by [3, Theorem 6.1, p. 202], $c_n(E) = 1$ for all n. Let T: $l_n^\infty \to H$ be an invertible linear operator from l_n^∞ onto a subspace of l_p.

Case I. Suppose $1 \leqslant p \leqslant 2$. By the proof of [7, Theorem 4.3, p. 289] $\pi_2(T) \leqslant K_G\|T\|$. Since $T^{-1}T$ is the identity operator on l_n^∞, $\pi_2(l_n^\infty) \leqslant \pi_2(T)\|T^{-1}\| \leqslant K_G\|T\|\,\|T^{-1}\|$. By [4, Theorem 10, p. 359], $n^{1/2} \leqslant \pi_2(l_n^\infty)$. Therefore, if E is a $\mathfrak{L}_{p,\lambda}$-space, then $c_n(E) \geqslant n^{1/2}/(\lambda K_G)$, and (c) follows.

Case II. Suppose $2 \leqslant p < \infty$. By the proof of [7, Theorem 8.2, p. 320],

$\pi_{p,2}(T) \leqslant K_G\|T\|$, where $\pi_{p,2}(T)$ is the $(p,2)$-absolutely summing norm of T. Since $T^{-1}T$ is the identity operator on l_n^∞, $\pi_{p,2}(l_n^\infty) \leqslant \pi_{p,2}(T)\|T^{-1}\| \leqslant K_G\|T\|\,\|T^{-1}\|$. But $n^{1/p} \leqslant \pi_{p,1}(l_n^\infty) \leqslant \pi_{p,2}(l_n^\infty)$. Therefore, if E is a $\mathcal{L}_{p,\lambda}$-space, then $c_n(E) \geqslant n^{1/p}/(\lambda K_G)$, and (b) holds.

THEOREM 2. *For a Banach space E the following are equivalent*:

(a) *E has the subspace factorization property.*

(b) *Every Schwartz space is linearly homeomorphic to a linear subspace of a compact projective limit of closed linear subspaces of E.*

(c) *Every Schwartz space is linearly homeomorphic to a linear subspace of some sufficiently high power E^I of E.*

(d) *E equipped with the topology of uniform convergence on null sequences in E' is a universal Schwartz space.*

(e) *Every compact linear operator between Banach spaces factors through a subspace of some finite power E^I of E.*

(f) *E contains l_n^∞ uniformly.*

PROOF. (a) implies (b). Let F be a Schwartz space. By [11, 2.2 and 2.3, p. 173], F is linearly homeomorphic to a linear subspace of a projective limit proj lim $f_{\mu\nu}(F_\nu)$ of Banach spaces $\{F_\nu: \nu \in I\}$, where each $f_{\mu\nu}: F_\nu \to F_\mu$ (for $\nu > \mu$ in I) is compact. Since E has the subspace factorization property, there is, for each $\nu > \mu$ in I, a closed linear subspace $E_{\mu\nu}$ of E and continuous linear operators $P_{\mu\nu}: F_\nu \to E_{\mu\nu}$ and $Q_{\mu\nu}: E_{\mu\nu} \to F_\mu$ such that $f_{\mu\nu} = Q_{\mu\nu}P_{\mu\nu}$. Let $J = \{(\mu,\nu): \mu > \nu \text{ in } I\}$. If (μ,ν) and (λ,δ) are in J, define $(\mu,\nu) > (\lambda,\delta)$ if and only if $\mu > \delta$ in I. J is then a directed set. Whenever $(\mu,\nu) > (\lambda,\delta)$ in J, let $g_{(\lambda,\delta)(\mu,\nu)} = P_{\lambda\delta}f_{\delta\mu}Q_{\mu\nu}$. It is easy to verify that the linear operator

$$T: \text{proj lim } f_{\mu\nu}(F_\nu) \to \text{proj lim } g_{(\lambda,\delta)(\mu,\nu)}(E_{\mu\nu})$$

defined by

$$T(\{x_\nu\}) = \{P_{\mu\nu}(x_\nu): (\mu,\nu) \in J\}$$

is a linear homeomorphism (into). Therefore, F is linearly homeomorphic to a linear subspace of the compact projective limit proj lim $g_{(\lambda,\delta)(\mu,\nu)}(E_{\mu\nu})$ of closed linear subspaces of E.

(b) implies (c) is obvious.

(b) implies (d). Let F be a Schwartz space. By (b) F is linearly homeomorphic to a compact projective limit proj lim $f_{\mu\nu}(E_\nu)$ of closed linear subspaces $\{E_\nu: \nu \in I\}$ of E. Let T denote the natural embedding of $G =$ proj lim $f_{\mu\nu}(E_\nu)$ into E^I and let S denote the identity operator from E^I into $E[\mathfrak{S}]^I$, where $\mathfrak{S}$ is the topology on E of uniform convergence on null sequences in E'. By [12, Proposition 1, p. 186], $E[\mathfrak{S}]$ is a Schwartz space. Since $\mathfrak{S}$ is weaker than the norm topology on E, S is continuous. To show that $E[\mathfrak{S}]$ is a universal Schwartz space it suffices to show that ST is open. Let $\mu \in I$ and let $U = \{x \in G: \|x_\mu\| \leqslant 1\}$. Choose $\nu > \mu$ so that $f_{\mu\nu}: E_\nu \to E_\mu$ is compact. By [15, Theorem 1, p. 76] and the Hahn-Banach theorem there is a null sequence $\{a_n\}$ in E' such that $\|f_{\mu\nu}(x)\| \leqslant \sup_n|\langle x,a_n\rangle|$ for every $x \in E_\nu$. Let $V = \{x \in E[\mathfrak{S}]^I: \sup_n|\langle x_\mu,a_n\rangle| \leqslant 1\}$. Since $V \cap ST(G) \subset ST(U)$, it follows that ST is open.

(c) implies (e). Let $T: F \to G$ be a compact linear operator between Banach spaces. Let $\mathfrak{S}$ denote the topology on F of uniform convergence on null sequences in F'. By [12, Proposition 1, p. 186], $F[\mathfrak{S}]$ is a Schwartz space and by [15, Theorem 1, p. 76], T: $F[\mathfrak{S}] \to G$ is continuous. By (c) there is a linear homeomorphism P: $F[\mathfrak{S}] \to E^I$ from the Schwartz space $F[\mathfrak{S}]$ into E^I; and there is a (unique) continuous linear operator S: $P(F[\mathfrak{S}]) \to G$ such that $T = SP$. By [2, Proposition 1, p. 40], there exists a finite subset J of I and continuous linear operators R: $P(F) \to E^J$ and Q: $RP(F) \to G$ such that $S = QR$. Since $\mathfrak{S}$ is weaker than the norm topology on F, RP: $F \to E^J$ is continuous (with respect to the norm topologies). Since $T = SP = Q(RP)$, it follows that T factors through a subspace of E^J.

(d) implies (e). Let T: $F \to G$ be a compact linear operator between Banach spaces. Let $\mathfrak{S}$ denote the topology on E of uniform convergence on null sequences in E'. By replacing E by $E[\mathfrak{S}]$ in the proof of "(c) implies (e)" above one can show that there is a finite set J such that T factors through a subspace of $E[\mathfrak{S}]^J$. That is, there exist continuous linear operators P: $F \to E[\mathfrak{S}]^J$ and Q: $P(F) \to G$ such that $T = QP$. Since F is barreled and $\mathfrak{S}$ is compatible with the dual system $\langle E,E' \rangle$, it is easy to see that P: $F \to E^J$ is continuous (with respect to the norm topologies). Since $\mathfrak{S}$ is weaker than the norm topology on E, Q: $P(F) \to G$ is continuous (with respect to the norm topologies). Therefore, $T = QP$ factors through a subspace of E^J.

(e) implies (f). For each positive integer n let $a_n = [\ln(n+1)]^{-1}$. Clearly, $a_n \to 0$ and $n^c a_n \to \infty$ for every $c > 0$. Since the linear operator $T: c_0 \to c_0$ defined by $T\lambda = \{a_n \lambda_n\}$ is compact, (e) implies that T factors through some finite power E^I of E. By [3, Theorem 6.1, p. 202], E^I contains l_n^∞ uniformly, whenever E^I is equipped with a norm that is compatible with the product topology of E^I. To complete the proof it suffices (using a simple induction argument) to show that:

Lemma 3. *If F and G are Banach spaces such that $F \times G$ contains l_n^∞ uniformly whenever $F \times G$ is equipped with a norm that is compatible with the product topology of $F \times G$, then either F or G contains l_n^∞ uniformly.*

Proof of Lemma 3. Let n be a fixed positive integer. Choose $\delta > 0$ so that $1 + \delta < (1 + n^{-1})(1 - 2^{n-1}\delta)$. Equip $F \times G$ with the norm $\|(f,g)\| = \sup(\|f\|, \|g\|)$. By assumption there exist points $(f_1,g_1), \ldots, (f_{2n},g_{2n})$ in $F \times G$ such that

$$\text{(i)} \qquad \|\lambda\|_\infty \leqslant \sup\left(\left\| \sum_{k=1}^{2n} \lambda_k f_k \right\|, \left\| \sum_{k=1}^{2n} \lambda_k g_k \right\| \right) \leqslant (1+\delta)\|\lambda\|_\infty .$$

whenever $\lambda = (\lambda_1, \ldots, \lambda_{2n})$ is a $2n$-tuple of scalars and $\|\lambda\|_\infty = \sup|\lambda_k|$. In particular, for each $1 \leqslant k \leqslant 2n$ either $\|f_k\| \geqslant 1$ or $\|g_k\| \geqslant 1$. Therefore, by interchanging F and G and by reindexing if necessary, we may assume that $\|f_k\| \geqslant 1$ for every $1 \leqslant k \leqslant n$. Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be an n-tuple of scalars and choose $1 \leqslant k \leqslant n$ so that $|\lambda_k| = \|\lambda\|_\infty = \sup|\lambda_j|$. By (i) it follows that

$$\text{(ii)} \qquad \|\epsilon_1 \lambda_1 f_1 + \cdots + \epsilon_n \lambda_n f_n\| \leqslant (1+\delta)\|\lambda\|_\infty .$$

whenever $(\epsilon_1, \ldots, \epsilon_n)$ is an n-tuple of plus or minus ones. Suppose that

$$\text{(iii)} \qquad \|\lambda_1 f_1 + \cdots + \lambda_n f_n\| < (1 - 2^{n-1}\delta)\|\lambda\|_\infty .$$

Since

$$2^{n-1}\lambda_k f_k = \bar{\Sigma}(\epsilon_1\lambda_1 f_1 + \cdots + \epsilon_n\lambda_n f_n)$$

where the sum is taken over the 2^{n-1} n-tuples $(\epsilon_1, \ldots, \epsilon_n)$ of plus or minus ones with $\epsilon_k = 1$, (ii) and (iii) imply that

$$\begin{aligned} 2^{n-1}\|\lambda\|_\infty &\leqslant \|2^{n-1}\lambda_k f_k\| \\ &< (1 - 2^{n-1}\delta)\|\lambda\|_\infty + (2^{n-1} - 1)(1 + \delta)\|\lambda\|_\infty \\ &\leqslant 2^{n-1}\|\lambda\|_\infty, \end{aligned}$$

a contradiction. Therefore, (iii) is false and

(iv) $$(1 - 2^{n-1}\delta)\|\lambda\|_\infty \leqslant \|\lambda_1 f_1 + \cdots + \lambda_n f_n\| \leqslant (1 + \delta)\|\lambda\|_\infty$$

whenever $\lambda = (\lambda_1, \ldots, \lambda_n)$ is an n-tuple of scalars. Let H denote the linear span of $\{f_1, \ldots, f_n\}$. Since $1 + \delta < (1 + n^{-1})(1 - 2^{n-1}\delta)$, (iv) implies that $d(H, l_n^\infty) < 1 + n^{-1}$. Therefore, either F or G contains "$(1 + n^{-1})$-copies" of l_n^∞ for arbitrarily large n, and the lemma follows.

The following corollary is a generalization of [11, Theorem 2.4, p. 173].

COROLLARY 4. *If E is an $\mathfrak{L}_\infty$-space* (*in the sense of* [5]), *then every Schwartz space is linearly homeomorphic to a linear subspace of a compact projective limit of E-spaces.*

PROOF. Let F be a Schwartz space. Since E contains l_n^∞ uniformly, Theorem 2 implies that F is linearly homeomorphic to a linear subspace of a compact projective limit proj lim $g_{\mu\nu}(E_\nu)$ of closed linear subspaces of E. Since E is an $\mathfrak{L}_\infty$-space, [8, Theorem 4.1, p. 336] implies that each $g_{\mu\nu}$: $E_\nu \to E_\mu$ can be extended to a compact linear operator $f_{\mu\nu}$: $E \to E$. It is now easy to verify that proj lim $g_{\mu\nu}(E_\nu)$ is linearly homeomorphic to a linear subspace of proj lim $f_{\mu\nu}(E)$.

COROLLARY 5. *If $\{E_\nu\}$ is a family of Banach spaces such that every Schwartz space is linearly homeomorphic to a linear subspace of some sufficiently high power $(\Pi E_\nu)^I$ of ΠE_ν, then at least one of the Banach spaces E_ν contains l_n^∞ uniformly.*

PROOF. This follows from the proof of Theorem 2 by making minor modifications in the proofs of "(c) implies (e)" and "(e) implies (f)".

COROLLARY 6. *If E is a locally convex space such that every Schwartz space is linearly homeomorphic to a linear subspace of some sufficiently high power E^I of E, then E locally contains l_n^∞ uniformly–that is, there is a fundamental system $\mathfrak{U}$ of closed balanced convex neighborhoods of* 0 *in E such that for each U in $\mathfrak{U}$ the associated normed space E_U* (*see* [5, *p.* 208]) *contains l_n^∞ uniformly.*

PROOF. Let V be a neighborhood of 0 in E and let $\mathfrak{U}_V$ denote the set of all closed balanced convex neighborhoods of 0 in E that are contained in V. To complete the proof it suffices to show that there is a $U \in \mathfrak{U}_V$ such that E_U contains l_n^∞ uniformly. Since $\mathfrak{U}_V$ is a fundamental system of neighborhoods of 0, it follows (from the proof of [14, 5.4, p. 53]) that E is linearly homeomorphic to a linear subspace of $\Pi\{E_U\colon U \in \mathfrak{U}_V\}$. By Corollary 5 there is a $U \in \mathfrak{U}_V$ such that the completion of E_U contains l_n^∞ uniformly. A simple density argument can now be used to show that E_U contains l_n^∞ uniformly.

Corollary 6 implies, in particular, that every universal Schwartz space locally contains l_n^∞ uniformly. This provides a partial answer to [11, Question 1, p. 175].

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