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A NOTE ON BOREL'S DENSITY THEOREM

HARRY FURSTENBERG

ABSTRACT. We prove the following theorem of Borel: *If G is a semisimple Lie group, H a closed subgroup such that the quotient space G/H carries finite measure, then for any finite-dimensional representation of G, each H-invariant subspace is G-invariant*. The proof depends on a consideration of measures on projective spaces.

The following is a relatively elementary proof of A. Borel's "density" theorem [1] (cf. also [5, Chapter V]). This theorem implies, among other things, that if Γ is a lattice subgroup of a connected semisimple real algebraic Lie group G with no compact factors, then Γ is Zariski dense in G. The main idea of the following proof can be found in [2, p. 347], but the connection with Borel's theorem escaped our notice.

Following von Neumann we call a locally compact topological group G *minimally almost periodic* (m.a.p.) if any continuous homomorphism of G into a compact group (equivalently, compact Lie group) is trivial [3]. The outstanding example of a m.a.p. group is a connected semisimple Lie group with no compact factors, but there are also discrete m.a.p. groups. (Cf. [3] and [4]. The semisimple case follows from the fact that the image of a semisimple group in a Lie group is closed.) Note that a m.a.p. group has no proper subgroups of finite index. Also a m.a.p. group is unimodular. In fact, any homomorphism of a m.a.p. group to the reals (indeed, to any abelian group) is trivial, since the reals have enough homomorphisms into compact groups to separate points.

Let V be a finite dimensional linear space. $P(V)$ will denote the corresponding projective space. If $v \in V$, $\bar{v}$ will denote the corresponding point of $P(V)$; if W is a subspace of V, $\overline{W}$ will designate the corresponding linear subvariety of $P(V)$. Finite unions of linear subvarieties will be called *quasi-linear* subvarieties. As for all algebraic subvarieties, these satisfy the descending chain condition. This leads to

LEMMA 1. *If A is a subset of $P(V)$, there exists a unique minimal quasi-linear subvariety $q(A) \subset P(V)$ containing A.*

A set of projective transformations is either relatively compact, or it is possible to extract a sequence of transformations that converges pointwise to

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a transformation that has as its range a quasi-linear subvariety of $P(V)$. A precise formulation is as follows.

LEMMA 2. *Let* $\tau_n \in GL(V)$ *and let* $\bar{\tau}_n$ *denote the corresponding projective transformations. Assume* $\det \tau_n = 1$ *and* $\|\tau_n\| \to \infty$, *where* $\| \ \|$ *is a suitable norm on the linear endomorphisms of* V. *There exists a transformation* π *of* $P(V)$ *whose range is a quasi-linear subvariety* $\subsetneqq P(V)$, *and a sequence* $\{n_k\}$ *with* $\bar{\tau}_{n_k}(x) \to \pi(x)$ *for every* $x \in P(V)$.

PROOF. Let W be any subspace of V. Passing to a subsequence and multiplying by appropriate constants, we can arrange that $\gamma_n \tau_n$ converges to a nonzero linear map of W into V. Calling this map σ_W we find that for $v \not\in \ker \sigma_W$, $\tau_n(\bar{v}) \to \overline{\sigma_W(v)}$ along the subsequence. Now inductively define subspaces of V by setting $W_0 = V$, $W_1 = \ker \sigma_{W_0}, \ldots, W_{i+1} = \ker \sigma_{W_i}, \ldots$, where each σ_{W_i} is the limit of successively finer and finer subsequences, appropriately normalised, of τ_n on the subspace W_i. Since $\dim W_{i+1} < \dim W_i$, the sequence of subspaces terminates. Now set $\pi(\bar{v}) = \overline{\sigma_{W_i}(v)}$ for $v \in W_i - W_{i+1}$. We will then have for some subsequence,

$$\tau_{n_k}(x) \to \pi(x), \quad \text{and} \quad \pi(P(V)) = \bigcup \overline{\sigma_{W_i}(W_i)}\,.$$

The proof of the lemma will be complete when we observe that $\det \sigma_V = 0$, so that $\sigma_{W_0} = \sigma_V$ is singular and its range is a proper subspace of V. For $i > 0$, the subspaces $\sigma_{W_i}(V)$ are clearly proper.

Our principal tool is the next lemma.

LEMMA 3. *Suppose* G *is a m.a.p. group. Let* $\rho\colon G \to GL(V)$ *be a representation of* G *on* V *and let* $\bar{\rho}$ *be the corresponding representation on* $P(V)$. *If* $\bar{\rho}(G)$ *preserves a finite measure* μ *on* $P(V)$, *then* μ *is concentrated on* $\bar{\rho}(G)$*-invariant points.*

PROOF. $\rho(G)$ cannot be relatively compact unless $\rho(G) = 1_V$. Assuming then that ρ is not trivial, we can find $g_n \in G$ with $\|\rho(g_n)\| \to \infty$. Passing to an appropriate subsequence, we can assume, according to the foregoing lemma, that the projective transformations $\bar{\rho}(g_n)$ converge pointwise to a transformation π whose range is a proper quasi-linear subvariety $Q \subset P(V)$. Now let $D(x)$ denote the distance, in some metric, from $x \in P(V)$ to the set Q. By the invariance of μ and the Lebesgue convergence theorem we have

$$\int D(x)\, d\mu(x) = \int D(\bar{\rho}(g_n)x)\, d\mu(x) \to 0.$$

It follows that μ is concentrated on the subvariety Q.

We now apply Lemma 1 and let X be the minimal quasi-linear subvariety containing the support of μ. $X \subset Q$ and so X is a proper subvariety of $P(V)$. Since $\bar{\rho}(G)$ preserves μ, it preserves its support, and so each $\bar{\rho}(g)$ must permute the components of X. There being only finitely many components in X, it follows that the elements of some subgroup of finite index in $\bar{\rho}(G)$ leave each component of X invariant. Since G has no proper subgroups of finite index, this is true for each $\bar{\rho}(g)$. Suppose now that W is a subspace of V with $\overline{W}$ a component of X. W is then an invariant subspace of $\rho(G)$. We could now repeat the entire argument for $\rho(G)|W$ and the restriction of μ to $\overline{W}$. If

$\rho(G)|W \neq 1_W$, we could find a proper quasi-linear subvariety of $\overline{W}$ that could replace $\overline{W}$ in X. Since this contradicts the definition of X, we must have $\rho(G)|X = 1$. This completes the proof of the lemma.

DEFINITION. A pair of groups (G, H) is called a *Borel pair* if G is a m.a.p. group and H is a closed subgroup of G such that the quotient space G/H supports a finite G-invariant measure.

In particular, if G is a semisimple Lie group which is connected and has no compact factors, and Γ is a lattice subgroup, then (G, Γ) is a Borel pair.

LEMMA 4. *Let (G, H) be a Borel pair, ρ: $G \to GL(V)$ a finite-dimensional representation of G, and assume that $\rho(H)$ leaves a 1-dimensional subspace $L \subset V$ invariant. Then $\rho(G)$ leaves L invariant.*

PROOF. Let $u \in P(V)$ correspond to L. Since $\bar{\rho}(H)u = u$, there is a continuous map π: $G/H \to P(V)$ with $\pi(gH) = \bar{\rho}(g)u$. π carries the invariant measure on G/H to an invariant measure on $P(V)$, and u is clearly in the support of this measure. By Lemma 3, u is fixed by $\bar{\rho}(G)$ and this proves the lemma.

THEOREM. *Let (G, H) be a Borel pair and ρ a finite-dimensional representation of G on a space V. If W is a $\rho(H)$-invariant subspace, it is also $\rho(G)$-invariant.*

PROOF. Let dim $W = r$ and form the exterior power $\Lambda^r\rho$ on $\Lambda^r V$. W corresponds to a 1-dimensional subspace of $\Lambda^r V$ and the foregoing lemma gives the desired result.

From this theorem we may deduce the remaining results of [1].

COROLLARY 1. *Let (G, H) be a Borel pair and let ρ be a representation of G. Then every matrix of $\rho(G)$ is a linear combination of matrices in $\rho(H)$.*

PROOF. The space spanned by $\rho(H)$ is $\rho(H)$-invariant. So it is $\rho(G)$-invariant. Since the identity matrix belongs to it so does all of $\rho(G)$.

COROLLARY 2. *Let (G, H) be a Borel pair and let ρ be a representation of G. Then the centralizer of $\rho(H)$ in $\rho(G)$ is the center of $\rho(G)$.*

This is clear by Corollary 1.

COROLLARY 3. *Let (G, H) be a Borel pair where G is a connected Lie group. Then the centralizer of H in G coincides with the center $Z(G)$ of G.*

PROOF. Consider $\rho =$ Ad and let $u \in$ centralizer of H in G. By the foregoing Ad u is in the center of Ad G and for any $g \in G$, $ugu^{-1}g^{-1} \in Z(G)$. Now one verifies that the map $g \to ugu^{-1}g^{-1}$ is a homomorphism of G into its center. But since G is m.a.p. this map is trivial. Hence $u \in Z(G)$.

Finally one proves the following, essentially as in [1].

COROLLARY 4. *Let (G, H) be a Borel pair where G is a connected Lie group. If L is a closed subgroup of G with finitely many components such that $L \supset H$, then $L = G$. In particular, if G is an algebraic group, then H is Zariski dense in G.*

PROOF. Let L_0 be the identity component of L, and let $\mathfrak{l}$ be its Lie algebra.

H normalizes L_0 so Ad H leaves $\mathfrak{l}$ invariant. By the theorem, Ad G leaves $\mathfrak{l}$ invariant so that L_0 is a normal subgroup of G. Since G preserves a finite measure on G/H and G/L is an equivariant image of G/H, G/L also possesses a G-invariant measure. Now G/L_0 is a finite covering space of G/L, and so it too has an invariant measure. But it is a group and so must be compact. Since G is m.a.p., $L_0 = G$, and so $L = G$.

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Institute of Mathematics, Hebrew University, Jerusalem, Israel