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# COMPACTNESS OF CERTAIN HOMOGENEOUS SPACES OF LOCALLY COMPACT GROUPS

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ABSTRACT. Let $H$ be the fixed points of a family of automorphisms of a locally compact group $G$ with $G/H$ finite invariant measure. It is proved in this paper that when the 1-component of $G$ is open, $G/H$ is compact.

Let $G$ be a locally compact group and $H$ be a closed subgroup of $G$ such that $G/H$ admits a finite $G$-invariant measure. Then $G/H$ is compact if $G$ is a connected Lie group and $H$ has finitely many connected components [6, Mostow], or if $G$ is a $p$-adic group and $H$ is discrete [8, Tamagawa]. Recently, Greenleaf-Moskowitz-Rothschild [1], [2] proved that $G/H$ is compact for disconnected Lie groups $G$ with $H$ consisting of the fixed points of a family of automorphisms of $G$ (see Lemma 3 below). Under similar restrictions on $H$ as in [1], the author [7] obtained the same result for linear algebraic groups defined over locally compact fields. Now in this paper, we prove the following theorem, which extends Lemma 3 to non-Lie groups.

THEOREM. *Let $G$ be a locally compact group and $H$ be a closed subgroup consisting of the fixed points of a family of automorphisms of $G$ such that $G/H$ has a finite $G$-invariant measure. If $G$ is $\sigma$-compact with its* 1*-component open then $G/H$ is compact. In particular, this is the case when $G$ is connected or when $G$ is $\sigma$-compact and locally connected.*

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1. **Preliminaries and notations.** Throughout this paper we consider only $\sigma$-compact groups (i.e. groups which are countable union of compact subsets). For a locally compact group $G$, let $G_0$ denote its 1-component, $\mathfrak{A}(G)$ the group of topological automorphisms of $G$ and $\mathfrak{I}(G) = \{\alpha_x | x \in G\}$ the subgroup of inner automorphisms of $G$. Let $K(G_0)$ denote the maximal compact normal subgroup of $G_0$ (the existence of such a group is proved in [4]). For a subset $A$ of $\mathfrak{A}(G)$, let $G_A = \{g \in G | \alpha(g) = g, \alpha \in A\}$. It is easy to see that $G_A$ is a closed subgroup of $G$.

A locally compact space $X$ is called a homogeneous $G$-space if $G$ acts on $X$ transitively. Thus when $H$ is a closed subgroup of $G$, $G/H$ is a homogeneous $G$-space with the action of $G$ on $G/H$ by left translation. A regular Borel measure $m$ on $X$ is $G$-invariant if $m(gE) = m(E)$ for all $g \in G$ and all Borel subsets $E$ of $X$.

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LEMMA 1. *Let $G$ and $G'$ be two locally compact groups and $X$ (resp. $X'$) be a homogeneous $G$-space (resp. $G'$-space). Let $\pi$: $G \to G'$ be an open and continuous epimorphism and $\eta$: $X \to X'$ be a continuous surjection such that $\eta(gx) = \pi(g)\eta(x)$ $(g \in G, x \in X)$.*

*We have the following*:

(i) *$\eta$ is an open map.*

(ii) *If $X$ admits a finite $G$-invariant measure, then $X'$ admits a finite $G'$-invariant measure.*

(iii) *If $\eta$ is bijective (i.e. if the actions of $G$ on $X$ and of $G'$ on $X'$ are equivalent), then*

(a) *the converse of* (ii) *holds, and*

(b) *$X$ is compact if and only if $X'$ is compact.*

PROOF. (i) Let $U$ be a neighborhood of some $x$ in $X$; we show that $\eta(U)$ contains an open neighborhood of $\eta(x)$. Since $G$ and $G'$ are $\sigma$-compact, it follows that the mappings

$$G \to X, \qquad G' \to X',$$
$$g \mapsto gx, \qquad g' \mapsto g'\eta(x),$$

are both open and continuous (see e.g. [3, p. 7]). Let $V$ be an open neighborhood of 1 in $G$ such that $Vx \subset U$. Hence $\pi(V)\eta(x) = \eta(Vx)$ is an open neighborhood of $\eta(x)$ contained in $\eta(U)$.

(ii) Let $m$ be a finite $G$-invariant measure on $X$. Define $m'$ on $X'$: $m'(E) = m(\eta^{-1}(E))$ for every Borel set $E$ of $X'$. Then $m'$ is a finite regular Borel measure on $X'$. Now for any $g' \in G'$, there exists a $g \in G$ such that $\pi(g) = g'$ and $\eta^{-1}(g'E) = g\eta^{-1}(E)$. Therefore

$$m'(g'E) = m(\eta^{-1}(g'E)) = m(g\eta^{-1}(E)) = m'(E).$$

(iii) (b) is obvious since $\eta$ is now a homeomorphism. For (a): Let $m'$ be a finite $G'$-invariant measure on $X'$ and define $m$ on $X$: $m(E) = m'(\eta(E))$ for any Borel subset $E$ of $X$. It is easy to see that $m$ is a finite $G$-invariant regular Borel measure.

LEMMA 2 [6, LEMMA 2.5]. *Let $H \subset F$ be closed subgroups of a locally compact group $G$ such that $G/H$ admits a finite invariant measure $m$. Then $G/F$ and $F/H$ admit finite invariant measures of which $m$ is a product.*

LEMMA 3 [2, THEOREM 2]. *If $G$ is a Lie group and $m(G/G_A)$ is finite, then $G/G_A$ is compact.*

LEMMA 4 [5, THEOREM 2.3]. *If $\pi$: $P \to P'$ is a continuous epimorphism of connected compact groups, then $\pi$ maps the center of $P$ onto the center of $P'$.*

2. **Proof of the theorem.** First we show that it suffices to consider the case when $G$ is connected. $G_0$ is open so $G_0 G_A$ is a closed subgroup of $G$. Hence, by Lemma 2, both $G/G_0 G_A$ and $G_0 G_A/G_A$ admit invariant measures. Since $G/G_0 G_A$ is discrete, $G/G_0 G_A$ is finite. On the other hand, the $G_0 G_A$-space $G_0 G_A/G_A$ is equivalent to the $G_0$-space $G_0/G_0 \cap G_A$ and it follows from Lemma 1 that $G_0/G_0 \cap G_A$ admits a finite invariant measure. But $G_0 \cap G_A$

$= (G_0)_{A'}$ where $A' = \{\alpha_{|G_0} | \alpha \in A\}$ is a subset of $\mathfrak{A}(G_0)$. And so by assumption $G_0/G_0 \cap G_A$ is compact and, by Lemma 1, $G_0 G_A/G_A$ is compact. Thus $G/G_A$ compact follows. This completes the proof of the reduction to the case when $G$ is connected.

From now on $G$ is connected and we proceed to prove the theorem in four cases.

*Case* (i). $K(G) = \{1\}$.

Let $P$ be a compact normal subgroup of $G$ such that $G/P$ is a Lie group. Since $PK(G)$ is a compact normal subgroup containing $K(G)$ and $K(G)$ is maximal, it follows that $P \subset PK(G) = K(G) = \{1\}$. Therefore $G$ is a Lie group. Thus by Lemma 3, $G/G_A$ is compact.

*Case* (ii). $K(G)_0 = \{1\}$.

Let $G' = G/K(G)$ and $\pi$: $G \to G'$ be the projection. For any $\alpha \in \mathfrak{A}(G)$, $\alpha(K(G))$ is again a compact normal subgroup of $G$ and so as in Case (i), $\alpha(K(G)) \subset K(G)$ (i.e. $K(G)$ is characteristic in $G$). Hence $\alpha$ induces an automorphism $\alpha'$ of $G'$ such that for any $g \in G$, $\alpha'(\pi(g)) = \pi(\alpha(g))$. Let $A' = \{\alpha' | \alpha \in A\}$ and $G'_{A'} = \{g' \in G' | \alpha'(g') = g', \alpha' \in A'\}$. Define a mapping

$$\eta\colon G/G_A \to G'/G'_{A'}, \quad \eta(gG_A) = \pi(g)G'_{A'}.$$

It is easy to see that $\eta$ is a continuous surjection and so, by Lemma 1, $G'/G'_{A'}$ has a finite $G'$-invariant measure. Since the pull back of any compact normal subgroup of $G'$ by $\pi$ is a compact normal subgroup of $G$ contained in $K(G)$, it follows that $K(G') = \{1\}$. Hence it follows from Case (i) that $G'/G'_{A'}$ is compact.

Let $H_A = \{g \in G | \alpha(g)g^{-1} \in K(G), \alpha \in A\}$. It is obvious that $H_A = \pi^{-1}(G'_{A'})$ is a closed subgroup of $G$. Define a mapping

$$\psi\colon G/H_A \to G'/G'_{A'}, \quad \psi(gH_A) = \pi(g)G'_{A'}.$$

Then it is easy to see that $\psi$ is a continuous bijection and so, by Lemma 1, $G/H_A$ is compact. Hence for $G/G_A$ to be compact, it remains to show that $H_A/G_A$ is compact.

Since $G_A \subset H_A$ and $G/G_A$ has a finite $G$-invariant measure, it follows from Lemma 2 that $H_A/G_A$ has a finite $H_A$-invariant measure. As $G$ is connected, it is obvious that $\mathfrak{I}(G)_{|K(G)} \subset \mathfrak{A}(K(G))_0$. Let $\mathfrak{I}^*(K(G)_0)$ denote the subgroup of inner automorphisms of $K(G)$ induced by elements of $K(G)_0$; then $\mathfrak{A}(K(G))_0 = \mathfrak{I}^*(K(G)_0)$ [4, Iwasawa]. Since $K(G)_0 = \{1\}$, $[G, K(G)] = \{1\}$. So for any $g_1, g_2$ in $H_A$, we have

$$\alpha(g_1 g_2)(g_1 g_2)^{-1} = \alpha(g_1)(\alpha(g_2)g_2^{-1})g_1^{-1} = \alpha(g_1)g_1^{-1}\alpha(g_2)g_2^{-1}.$$

Hence the mapping $f$: $H_A \to K(G)$, $f(g) = \alpha(g)g^{-1}$ is a homomorphism with $G_A$ as its kernel. Hence $G_A$ is normal in $H_A$ and $H_A/G_A$ is compact. This completes the proof of Case (ii).

*Case* (iii). The center $Z$ of $K(G)_0$ is trivial.

Since $K(G)_0$ is characteristic in $K(G)$ and $K(G)$ is characteristic in $G$, it follows that $K(G)_0$ is characteristic in $G$. Let $G' = G/K(G)_0$ and $\pi$: $G \to G'$ be the projection. Then as in Case (ii) $A$ induces a family $A'$ of automorphisms of $G'$ such that $G'/G'_{A'}$ admits a finite $G'$-invariant measure. Since it is easy to see that $(K(G'))_0 = \{1\}$, it follows from Case (ii) that $G'/G'_{A'}$ is compact.

Let $C = \{g \in G | gkg^{-1} = k, k \in K(G)_0\}$; then it follows from [4] that $G = K(G)_0 C$. Here $K(G)_0 \cap C = Z = \{1\}$ and $[K(G)_0, C] = \{1\}$. And so $G = K(G)_0 \times C$ is a direct product and $\pi_{|C}\colon C \to G'$ is an isomorphism. Now let $C_A = \{c \in C | \alpha(c) = c, \alpha \in A\}$. We shall show that $\pi(C_A) = G'_{A'}$. It is obvious that $\pi(C_A) \subset G'_{A'}$. To see the converse inclusion, let $g' \in G'_{A'}$; then there is a unique $c \in C$ such that $\pi(c) = g'$. So for any $\alpha \in A$, $\pi(c) = \alpha'(\pi(c)) = \pi(\alpha(c))$ or $\alpha(c)c^{-1} \in K(G)_0$. But $C$ is characteristic in $G$, hence $\alpha(c)c^{-1} \in C \cap K(G)_0$. Thus $\alpha(c) = c$ or $g' \in \pi(C_A)$. Therefore $\pi_{|C}$ induces a homeomorphism $\eta\colon C/C_A \to G'/G'_{A'}$, $\eta(cC_A) = \pi(c)G'_{A'}$. Hence $C/C_A$ is compact.

Let $K_A = \{k \in K(G)_0 | \alpha(k) = k, \alpha \in A\}$ and define

$$\psi\colon (K(G)_0/K_A) \times (C/C_A) \to G/G_A,\ \psi(kK_A, cC_A) = kcG_A.$$

Here $\psi$ is well defined, for if $kk_1^{-1} \in K_A$, $cc_1^{-1} \in C_A$, then $(kc)(k_1c_1)^{-1} = (kk_1^{-1})(cc_1^{-1}) \in K_A C_A \subset G_A$.

Let $\psi_1\colon G \to G/G_A$ be the continuous projection and $\psi_2\colon K(G)_0 \times C \to (K(G)_0/K_A) \times (C/C_A)$ be the open projection. Then $\psi_1 = \psi \circ \psi_2$ since $G = K(G)_0 \times C$. Hence $\psi$ is continuous and $G/G_A$ is compact.

*Case* (iv). $Z \neq \{1\}$.

Since $K(G)_0$ is characteristic in $G$, so is $Z$. Let $G' = G/Z$ and $\pi\colon G \to G'$ be the projection. Then as in Case (ii) $A$ induces a family $A'$ of automorphisms of $G'$ such that $G'/G'_{A'}$ admits a finite $G'$-invariant measure. As $Z$ is compact, therefore $\pi(K(G)_0) = (K(G'))_0$. Hence by Lemma 4, we have center of $(K(G'))_0 = \pi(Z)$. But $\pi(Z) = \{1\}$, therefore it follows from Case (iii) that $G'/G'_{A'}$ is compact.

Let $H_A = \{g \in G | \alpha(g)g^{-1} \in Z, \alpha \in A\}$. Then $H_A = \pi^{-1}(G'_{A'})$ is a closed subgroup of $G$ containing $G_A$. As in Case (ii), we have $G/H_A$ compact. So for $G/G_A$ to be compact, it remains to prove that $H_A/G_A$ is compact. Now

$$\mathfrak{I}(G)_{|Z} \subset \mathfrak{A}(Z)_0 = \mathfrak{I}^*(Z_0) = \{1\}$$

where $\mathfrak{I}^*(Z_0)$ denotes the subgroup of inner automorphisms of $Z$ induced by elements of $Z_0$. Therefore $[Z, G] = \{1\}$ and analogous arguments as those in Case (ii) show that $H_A/G_A$ is compact. This completes the proof of the theorem.

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