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AN ANALOGUE OF SOME INEQUALITIES OF P. TURAN CONCERNING ALGEBRAIC POLYNOMIALS HAVING ALL ZEROS INSIDE $[-1, +1]$

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ABSTRACT. Let $P_n(x)$ be an algebraic polynomial of degree $\leqslant n$ having all its zeros inside $[-1, +1]$; then we have

$$\int_{-1}^{1} P_n'^2(x)\,dx > (n/2)\int_{-1}^{1} P_n^2(x)\,dx.$$

The result is essentially best possible. Other related results are also proved.

Let H_n be the set of all polynomials whose degree does not exceed n, i.e., polynomials of the form

$$P(x) = c_0 + c_1x + c_2x^2 + \cdots + c_nx^n.$$

Here the coefficients $c_0, c_1, \ldots, c_n$ are arbitrary real numbers. The following inequalities on algebraic polynomials are well known.

THEOREM A. *Let $P_n(x) \in H_n$; then we have*

$$\max_{-1 \leqslant x \leqslant +1} (1 - x^2)P_n'^2(x) \leqslant n^2 \max_{-1 \leqslant x \leqslant +1} P_n^2(x), \tag{1.1}$$

and

$$\max_{-1 \leqslant x \leqslant +1} |P_n'(x)| \leqslant n^2 \max_{-1 \leqslant x \leqslant +1} |P_n(x)|. \tag{1.2}$$

($P_n'(x)$ stands for the derivative of $P_n(x)$.)

(1.1) is due to S. N. Bernstein [1] and (1.2) to A. A. Markov [2]. In this work we are concerned with the following beautiful theorem of P. Turán [4].

THEOREM B. *Let $P_n(x)$ be an algebraic polynomial of degree $\leqslant n$ having all its zeros inside $[-1, +1]$; then we have*

$$\max_{-1 \leqslant x \leqslant +1} |P_n'(x)| > \frac{n^{1/2}}{6} \max_{-1 \leqslant x \leqslant 1} |P_n(x)|. \tag{1.3}$$

(1.3) was later sharpened by Janos Eröd [2], who proved

THEOREM C. *Under the assumptions of Theorem* B *we have for $P_n(x) \in H_n$*

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$$\frac{\max_{-1\leqslant x\leqslant +1}|P_n'(x)|}{\max_{-1\leqslant x\leqslant +1}|P_n(x)|} \geqslant \frac{n}{2} \quad \text{if } n = 2, 3,$$

$$(1.4) \qquad \geqslant \frac{n}{\sqrt{n-1}}\left(1-\frac{1}{n-1}\right)^{(n-2)/2}, \qquad n \text{ even}, n \geqslant 4,$$

$$\geqslant \frac{n^2}{(n-1)\sqrt{n+1}}\left(1-\frac{\sqrt{n+1}}{n-1}\right)^{(n-3)/2}\left(1+\frac{1}{\sqrt{n+1}}\right)^{(n-1)/2},$$

if $n \geqslant 5$ and odd.

Further, this result is best possible.

Inequalities on polynomials analogous to (1.2) in the norm

$$\|f\|^2_{L_2[-1,+1]} = \int_{-1}^{1} f^2(x)\,dx$$

were proved by E. Schmidt [5]. See also the contributions of Einar Hille, G. Szegö and J. D. Tamarkin [4].

THEOREM D [E. SCHMIDT]. *Let us denote*

$$(1.5) \qquad M_n^2 = \max_{f\in H_n}\left[\int_{-1}^{+1} f'^2(x)\,dx \Big/ \int_{-1}^{+1} f^2(x)\,dx\right];$$

then for $n \geqslant 5$ $(-6 < R < 13)$

$$(1.6) \qquad M_n = \frac{(n+\frac{3}{2})^2}{\pi}\left(1-\frac{\pi^2-3}{12(n+\frac{3}{2})^2}+\frac{R}{(n+\frac{3}{2})^4}\right)^{-1}.$$

In view of Theorems B and D it is natural to ask: If $P_n \in H_n$ and all zeros of $P_n(x)$ are inside $[-1, +1]$, then how small can the expression $\int_{-1}^{+1} P_n'^2(x)\,dx / \int_{-1}^{+1} P_n^2(x)\,dx$ be? The following theorem concerns the above question.

THEOREM 1. *Let $P_n(x) \in H_n$ and assume that all its zeros are inside $[-1, +1]$; then we have*

$$(1.7) \qquad \int_{-1}^{+1} P_n'^2(x)\,dx > \frac{n}{2}\int_{-1}^{+1} P_n^2(x)\,dx.$$

This result is best possible in the sense that there exists a polynomial $p_0(x)$ of degree n having all zeros inside $[-1, +1]$ and for which

$$(1.8) \quad \int_{-1}^{+1} P_0'^2(x)\,dx \Big/ \int_{-1}^{+1} P_0^2(x)\,dx = \frac{n}{2}+\frac{3}{4}+\frac{3}{4(n-1)}, \qquad n > 1.$$

The proof of Theorem 1 is based on

THEOREM 2A. *Let $f_n(x)$ be an algebraic polynomial of degree $\leqslant n$ having all zeros inside $[-1, +1]$; then we have*

$$(1.9) \qquad \frac{n}{2} \leqslant \int_{-}^{+1} f_n'^2(x)(1-x^2)\,dx \Big/ \int_{-1}^{+1} f_n^2(x)\,dx,$$

with equality only for $f_n(x) = (1+x)^p(1-x)^q$, $p+q=n$.

THEOREM 2B. *Let $f_n(x)$ be an algebraic polynomial of degree $\leqslant n$; then we have*

$$\int_{-1}^{+1} f_n'^2(x)(1-x^2)\,dx \Big/ \int_{-1}^{+1} f_n^2(x)\,dx \leqslant n(n+1),$$

with equality only for $f_n(x) = cP_n(x)$ ($P_n(x)$ being the Legendre polynomial of degree n).

(1.10) may be regarded as analogous to (1.1) in the L_2 norm.

2. Since $f_n(x)$ has all zeros inside $[-1, +1]$ we may write

$$f_n(x) = \prod_{k=1}^{n} (x - x_k) \tag{2.1}$$

where $-1 \leqslant x_k \leqslant +1$, $k = 1, 2, \ldots, n$.

Professor P. Turán [6] observed that

$$f_n'(x) = f_n(x) \sum_{k=1}^{n} \frac{1}{(x - x_k)} \tag{2.2}$$

and

$$f_n'^2(x) - f_n(x) f_n''(x) = f_n^2(x) \sum_{k=1}^{n} \frac{1}{(x - x_k)^2}. \tag{2.3}$$

On multiplying (2.3) by $(1 - x^2)$ and using (2.2) we obtain

$$\begin{aligned} 2(1-x^2) f_n'^2(x) &- \frac{d}{dx}\left((1-x^2) f_n(x) f_n'(x)\right) \\ &= (1-x^2) f_n^2(x) \sum_{k=1}^{n} \frac{1}{(x-x_k)^2} + 2x f_n(x) f_n'(x) \\ &= f_n^2(x) \sum_{k=1}^{n} \frac{(1-x^2) + 2x(x-x_k)}{(x-x_k)^2}. \end{aligned}$$

Therefore

$$\begin{aligned} 2(1-x^2) f_n'^2(x) - n f_n^2(x) &- \frac{d}{dx}\left\{(1-x^2) f_n(x) f_n'(x)\right\} \\ &= f_n^2(x) \sum_{k=1}^{n} \frac{1 - x^2 + 2x(x-x_k) - (x-x_k)^2}{(x-x_k)^2} \\ &= f_n^2(x) \sum_{k=1}^{n} \frac{(1-x_k^2)}{(x-x_k)^2} \geqslant 0. \end{aligned} \tag{2.4}$$

On integrating both sides from -1 to 1 we obtain

$$2\int_{-1}^{+1} (1-x^2) f_n'^2(x)\,dx - n\int_{-1}^{+1} f_n^2(x)\,dx \geqslant 0, \tag{2.5}$$

with equality only for $f_n(x) = (1+x)^p(1-x)^q$, $p + q = n$. But this in turn implies (1.9). This proves Theorem 2A.

The proof of Theorem 2B depends on

$$\int_{-1}^{+1} p_i(x) p_j(x)\, dx = 0, \qquad i \neq j, \tag{2.6}$$
$$= 2/(2i+1), \qquad i = j,$$

and

$$\int_{-1}^{+1} (1-x^2) P_i'(x) P_j'(x)\, dx = 0, \qquad i \neq j, \tag{2.7}$$
$$= 2i(i+1)/(2i+1), \qquad i = j,$$

where $P_j(x)$ denotes the Legendre polynomial of degree j in x. Writing

$$f_n(x) = \sum_{i=0}^{n} \lambda_i P_i(x), \qquad f_n'(x) = \sum_{i=1}^{n} \lambda_i P_i'(x)$$

and using (2.6) and (2.7) we obtain

$$\frac{\int_{-1}^{+1} f_n'^2(x)(1-x^2)\, dx}{\int_{-1}^{+1} f_n^2(x)\, dx} = \frac{\sum_{i=1}^{n} \lambda_i^2 i(i+1)/(2i+1)}{\sum_{i=0}^{n} \lambda_i^2/(2i+1)} \leqslant n(n+1).$$

Obviously equality occurs only if $f_n(x) = cP_n(x)$ ($P_n(x)$ being Legendre polynomial of degree n). This proves Theorem 2B.

PROOF OF THEOREM 1. Since $1 - x^2 \leqslant 1$, $-1 \leqslant x \leqslant +1$, we have

$$(1-x^2) P_n'^2(x) \leqslant P_n'^2(x) \quad \text{for } -1 \leqslant x \leqslant +1.$$

Therefore

$$\int_{-1}^{+1} (1-x^2) P_n'^2(x)\, dx \leqslant \int_{-1}^{+1} P_n'^2(x)\, dx. \tag{2.8}$$

But from Theorem 2A we have

$$\int_{-1}^{+1} (1-x^2) P_n'^2(x)\, dx \geqslant \frac{n}{2} \int_{-1}^{+1} P_n^2(x)\, dx. \tag{2.9}$$

From (2.8) and (2.9) it follows that

$$\int_{-1}^{+1} P_n'^2(x)\, dx \geqslant \frac{n}{2} \int_{-1}^{+1} P_n^2(x)\, dx.$$

This proves (1.7). It remains to prove (1.8). Let $f_n(x) = p_0(x) = (1-x^2)^m$, $2m = n$; then

$$\int_{-1}^{1} p_0^2(x)\, dx = \int_{-1}^{+1} (1-x^2)^m\, dx = \frac{2\Gamma(n+1)\Gamma(\frac{1}{2})}{\Gamma(n+\frac{3}{2})}$$

and

$$\int_{-1}^{1} P_0'^2(x)\, dx = n^2 \int_{-1}^{1} (1-x^2)^{n-2} x^2\, dx$$
$$= 4n^2 \int_0^{\pi/2} \sin^{2n-3}\theta \cos^2\theta\, d\theta$$
$$= n^2 \frac{\Gamma(n-1)\Gamma(\frac{1}{2})}{\Gamma(n+\frac{1}{2})}.$$

Therefore

$$\frac{\int_{-1}^{1} P_0'^2(x)\,dx}{\int_{-1}^{1} P_0^2(x)\,dx} = \frac{n(n+\frac{1}{2})}{2(n-1)} = \frac{n}{2} + \frac{3}{4} + \frac{3}{4(n-1),}, \qquad n > 1.$$

This proves Theorem 1 as well.

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