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IRREDUCIBLES IN THE LANDWEBER NOVIKOV ALGEBRA

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ABSTRACT. All the irreducible and reducible elements in the Landweber Novikov algebra are determined. A full set of relations mod reducibles is given.

1. **Introduction.** Let S^* denote the Landweber Novikov algebra, and let $\bar{S}$ be the kernel of the augmentation map. The aim of this paper is to compute $Q(S^*) = \bar{S}/\bar{S}^2$, the module of irreducibles.

For every exponent sequence α with only finitely many nonzero terms, Landweber [1] and Novikov [2] define an operation $s_\alpha \in S^*$. Moreover, the s_α's form a basis for S^* as a Z-module.

For every exponent sequence $\alpha = (a_1, \ldots, a_n, \cdots)$, let $\|\alpha\| = \Sigma i a_i$, and $|\alpha| = \Sigma a_i$. Let $\Delta(a)$ denote the exponent sequence all of whose elements are zero except 1 in the ath place. Our main theorem is

THEOREM 1.1. (a) *$Q(S^*)$ is generated by $\{s_{p^n\Delta(1)}, s_{p^n\Delta(2)} | p$ prime, $n \geqslant 0\}$, with the only relations $ps_{p^n\Delta(1)} \in \bar{S}^2$ for $n \geqslant 2$ and every p, $ps_{p\Delta(1)} \in \bar{S}^2$ for $p \neq 2$, $ps_{p^n\Delta(2)} \in \bar{S}^2$ for $n \geqslant 1$ and $2(s_{\Delta(2)} + s_{2\Delta(1)}) \in \bar{S}^2$.*

(b) *All the s_α's are reducible except for $\alpha = p^n\Delta(1)$, $p^n\Delta(2)$, $2p^n\Delta(1)$. The only relations between irreducibles are those specified in* (a) *and $s_{p^n\Delta(2)} + s_{2p^n\Delta(1)} \in \bar{S}^2$ for $p \neq 2$ and $n > 0$.*

Our main computational tool is the following theorem due to Landweber [1].

Let S_* be the dual algebra to S^*. Let σ_α be the dual basis to s_α. Then S_* is a polynomial algebra with generators $\{\sigma_{\Delta(a)}\}_{a \geqslant 1}$ and

THEOREM 1.2. *The diagonal in S_* is given by*

$$\phi_*(\sigma_{\Delta(a)}) = \sum_{\|\alpha\| + i = a} \binom{i+1}{\alpha} \sigma_\alpha \otimes \sigma_{\Delta(i)}.$$

ADDED IN PROOF. While writing this paper I heard that Aikawa [3] got the same results. I would like to thank Shibata for reading this paper and correcting many of the mistakes appearing in the original version.

2. **Definition.** If $\alpha = (a_1, \ldots, a_n, \ldots)$, let $\nu_p(\alpha) = \min_i\{\nu_p(a_i)\}$ and $\nu_p(ns_\alpha) = \max\{0, \nu_p(\alpha) - \nu_p(n)\}$.

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Let us say that two exponent sequences $\alpha = (a_1, \ldots, a_n, \ldots)$, $\beta = (b_1, \ldots, b_n, \ldots)$ are disjoint if $a_i b_i = 0$ for every i.

THEOREM 2.1. (a) *For every* α,β *we have* $s_\alpha \circ s_\beta = \lambda s_{\alpha+\beta} + \Sigma\lambda_i s_{\alpha_i}$ *where* $|\alpha_i| < |\alpha + \beta|$. *If* α *and* β *are disjoint, then* $\lambda = 1$. *Moreover,* $\nu_p(\lambda_i s_{\alpha_i}) \leqslant \min\{\nu_p(\alpha), \nu_p(\beta)\}$.

(b) *For* $n > 1$ *there exists a* $\lambda \in Z$ *such that* $\lambda s_{n\Delta(a)} \equiv \Sigma\lambda_i s_{\alpha_i} \bmod \bar{S}^2$, *where* $|\alpha_i| < n$ *and* $\nu_q(\lambda_i s_{\alpha_i}) \leqslant \nu_q(n)$ *for every prime* q. *Moreover,* $\lambda = 1$ *if* n *is not a power of a prime, and* $\lambda = p$ *if* $n = p^k$ *for some prime* p.

PROOF OF (a). We will prove (a) by passing to the dual. That is, if $\phi_*(\sigma_\gamma) = \lambda\sigma_\alpha \otimes \sigma_\beta + \cdots$ with $\lambda \neq 0$, then $|\gamma| < |\alpha + \beta|$ unless $\gamma = \alpha + \beta$. This will follow from 1.2 by trivial induction on $|\gamma|$. We also have to show that $\min\{\nu_p(\alpha), \nu_p(\beta)\} \geqslant \nu_p(\gamma) - \nu_p(\lambda)$. Let $r = \nu_p(\gamma)$, i.e. $\gamma = p^r\delta$. Hence,

$$\phi_*(\sigma_\gamma) = \phi_*(\sigma_\delta)^{p^r} = \left(\sum \mu_i \sigma_{\alpha_i} \otimes \sigma_{\beta_i}\right)^{p^r},$$

and we will get the results from the following lemma.

LEMMA 2.2. *If* $(\Sigma y_i)^{p^r} = \Sigma\lambda_i z_i$, *where the* z_i*'s are monomials in the* y_i*'s, then* $\nu_p(z_i) + \nu_p(\lambda_i) \geqslant r$, *where* $\nu_p(z) = \max\{t \mid \exists y \text{ with } z = y^{p^t}\}$.

PROOF OF 2.1 (b). From (a) we have that if $k + l = n$, then

$$s_{k\Delta(a)} \circ s_{l\Delta(a)} = \binom{n}{k} s_{n\Delta(a)} + \sum \lambda_i s_{\alpha_i}$$

where $|\alpha_i| < n$ and $\nu_p(\lambda_i s_{\alpha_i}) \leqslant \nu_p(n)$ for every prime p. But g.c.d. $\{\binom{n}{k}\}$ is the same λ defined in the theorem, and, hence, we can take an appropriate linear combination of the above relations to get (b).

COROLLARY 2.3. *For every* n *and* α *we have that*

$$ns_\alpha \equiv \sum_{p,a,i} \lambda_{p,a,i} s_{p^i\Delta(a)} \mod \bar{S}^2$$

where $i \leqslant \nu_p(ns_\alpha)$.

PROOF. The proof is by induction on $|\alpha|$. If α is not of the form $m\Delta(a)$, then there are disjoint β,γ such that $\alpha = \beta + \gamma$. Then by 2.1(a), $ns_\alpha \equiv \Sigma n\lambda_i s_{\alpha_i}$ with $\nu_p(n\lambda_i s_{\alpha_i}) \leqslant \nu_p(ns_\alpha)$ and $|\alpha_i| < |\alpha|$. Apply now the induction hypothesis to α_i and $n\lambda_i$.

If $\alpha = m\Delta(a)$, do the same using 2.1(b).

LEMMA 2.4. *For every* $a \neq b$,

(a) $$s_{\Delta(a)} \circ s_{\Delta(b)} = s_{\Delta(a)+\Delta(b)} + (b+1)s_{\Delta(a+b)}.$$

(b) $$s_{\Delta(a)} \circ s_{2\Delta(b)} = \lambda s_{\Delta(a)+2\Delta(b)} + (b+1)s_{\Delta(b)+\Delta(a+b)}.$$

(c) $$s_{2\Delta(b)} \circ s_{\Delta(a)} = \lambda s_{\Delta(a)+2\Delta(b)} + (a+1)s_{\Delta(b)+\Delta(a+b)} + \binom{a+1}{2} s_{\Delta(a+2b)}.$$

λ *is the same as in* b *and* $\lambda = 1$ *if* $a \neq b$.

(d) $s_{\Delta(a)} \equiv 0 \bmod \bar{S}^2$ *for every* $a \neq 1,2$.

PROOF. (a), (b), (c) are routine computations. To prove (d) we will have to separate cases:

(1) a odd, $a \neq 1$. Write $a = b + c$ with $b - c = 1$. Then by (a), $[s_{\Delta(b)}, s_{\Delta(c)}] = (b - c)s_{\Delta(a)} = s_{\Delta(a)}$.

(2) a even, $a \neq 2$. Write $a = b + c$ with $b - c = 2$. Then as in (1) we get $2s_{\Delta(a)} \equiv 0 \bmod \bar{S}^2$.

Let $a = 2 + 2b$. Using (b), (c) and (a) we get that

$$\left[s_{\Delta(2)}, s_{2\Delta(b)}\right] = (b - 2)s_{\Delta(b)+\Delta(b+2)} + 3s_{\Delta(a)}$$

and

$$s_{\Delta(b+2)} \circ s_{\Delta(b)} = s_{\Delta(b)+\Delta(b+2)} + (b + 1)s_{\Delta(a)}.$$

Combining both we get

$$s_{\Delta(a)} \equiv [(b + 1)(b - 2) - 2]s_{\Delta(a)} \bmod \bar{S}^2.$$

But $(b + 1) \cdot (b - 2)$ is even and, hence, $s_{\Delta(a)} \in \bar{S}^2$.

LEMMA 2.5.

(a)
$$s_{p^n\Delta(a)} \circ s_{p^n\Delta(b)} \equiv (b + 1)^{p^n} s_{p^n\Delta(a+b)} + s_{p^r(\Delta(a)+\Delta(b))} + \sum_{c;k<n} \lambda_{c,k} s_{p^k\Delta(c)} \bmod \bar{S}^2 \quad \textit{for } a \neq b.$$

(b)
$$s_{p^n\Delta(a)} \circ s_{2p^n\Delta(b)} \equiv \lambda^{p^n} s_{p^n(\Delta(a)+2\Delta(b))} + (b + 1)^{p^n} s_{p^n(\Delta(b)+\Delta(a+b))} + \sum_{c;k<n} \lambda_{c,k} s_{p^k\Delta(c)} \bmod \bar{S}^2.$$

(c)
$$s_{2p^n\Delta(b)} \circ s_{p^n\Delta(a)} \equiv \lambda^{p^n} s_{p^n(\Delta(a)+2\Delta(b))} + (a + 1)^{p^n} s_{p^n(\Delta(b)+\Delta(a+b))} + \binom{a+1}{2}^{p^n} s_{p^n\Delta(a+2b)} + \sum_{c;k<n} \lambda_{c,k} s_{p^k\Delta(c)} \bmod \bar{S}^2.$$

The constant λ *appearing in* (b) *and* (c) *is the same* λ *as in* 2.4.

(d) *For every* $a \neq 1,2$ *we have* $s_{p^n\Delta(a)} \in \bar{S}^2$.

PROOF. The proof of (a), (b) and (c) are identical, so we will prove (b). By 2.1 we have that

$$s_{p^n\Delta(a)} \circ s_{2p^n\Delta(b)} = \lambda^{p^n} s_{p^n(\Delta(a)+2\Delta(b))} + \sum \lambda_i s_{\alpha_i}$$

with $|\alpha_i| < 3p^n$, and that for every prime $q \neq p$, $\nu_q(\lambda_i s_{\alpha_i}) = 0$. We want to show that the only possible α_i in the sum with $\nu_p(\alpha_i) \geqslant n$ is $p^n(\Delta(b) + \Delta(a + b))$. This will imply (b) by Corollary 2.3.

But if $\alpha_i = p^n\beta$ and $|\alpha_i| \leqslant 2p^n$, then β must be of the form $\Delta(t)$ or $\Delta(t) + \Delta(s)$ or $2\Delta(t)$. An immediate check leaves the only possibility $\beta = \Delta(a) + \Delta(a + b)$.

(d) The proof is by induction on n, the case $n = 0$ having been done in 2.4(d). For $n > 0$ one follows the proof of 2.4(d). The only extra fact which is needed is that $ps_{p^n\Delta(a)} \in \bar{S}^2$ for $a \geqslant 2$, but this will follow from $ps_{p^n\Delta(a)}$
43 $\equiv \sum_{i<n;b} \lambda_{i,b} s_{p^i\Delta(b)}$ and our induction hypothesis. (Note that in the above sum we have $b > 2$, so induction applies.)

PROOF OF THEOREM 1.1. Let $\|\alpha\| = n$ with n not of the form p^m or $2p^m$. Then $s_\alpha \equiv \sum_{p,i,a} \lambda_{p,i,a} s_{p^i\Delta(a)} \bmod \bar{S}^2$ where $a \neq 1,2$. Then by 2.5(d), s_α is reducible.

If $\|\alpha\| = p^n$ or $2p^n$, but $\alpha \neq p^n\Delta(1), p^n\Delta(2)$ or $2p^n\Delta(1)$, then $\nu_p(\alpha) < n$ and $s_\alpha \equiv \sum_{i,a}\lambda_{i,a}s_{p^i\Delta(a)}$ with $a \neq 1,2$ and, as before, s_α is reducible.

As for the remaining cases, we have already shown in the proof of 2.5(d) that $ps_{p^n\Delta(a)} \in \bar{S}^2$ for $a \geqslant 2$. The same proof works if $a = 1$ and $p > 2$. To show that these are the only relations, look at:

$\phi_*(\sigma_{p^n\Delta(1)}) = (\sigma_{\Delta(1)} \otimes 1 + 1 \otimes \sigma_{\Delta(1)})^{p^n}$. So in any relation where $s_{p^n\Delta(1)}$ appears, it is with coefficient divisible by p. Hence, it is irreducible.

$\phi_*(\sigma_{p^n\Delta(2)}) = (\sigma_2 \otimes 1 + 2\sigma_1 \otimes \sigma_1 + 1 \otimes \sigma_2)^{p^n}$. Hence, the only relation in which $s_{p^n\Delta(2)}$ appears with a coefficient which is not divisible by p is

$$s_{p^n\Delta(1)} \circ s_{p^n\Delta(1)} = 2^{p^n}s_{p^n\Delta(2)} + \binom{2p^n}{p^n}s_{2p^n\Delta(1)} + \cdots .$$

Similarly, the previous relation is the only interesting one for $s_{2p^n\Delta(1)}$. The other terms in this expression are all in $\bar{S}^2$.

If $p > 2$ we have

$$\binom{2p^n}{p^n} \equiv 2^{p^n} \equiv 2 \mod p.$$

Hence, $2(s_{2p^n\Delta(1)} + s_{2p^n\Delta(2)}) \in \bar{S}^2$. We also have $p(s_{2p^n\Delta(1)} + s_{p^n\Delta(2)}) \in \bar{S}^2$ and we get our theorem for $p \neq 2$. If $p = 2$ we have

$$\binom{2p^n}{p^n} \equiv 2 \mod 4.$$

Hence, $2s_{2^{n+1}\Delta(1)} \in \bar{S}^2$ for every $n \geqslant 1$, which finishes our proof.

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