PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 55, Number 2, March 1976

ON PROPERLY EMBEDDING PLANES IN 3-MANIFOLDS

E. M. BROWN, M. S. BROWN AND C. D. FEUSTEL[1]

ABSTRACT. In this paper we prove an analog of the loop theorem for a certain class of noncompact 3-manifolds. In particular, we show that the existence of a "nontrivial" proper map of a plane into a 3-manifold implies the existence of a nontrivial proper embedding of a plane into a 3-manifold.

Introduction. In this paper we prove an analog of the loop theorem for a certain class of noncompact 3-manifolds. More precisely we show that the existence of a "nontrivial" proper map of a plane into a noncompact eventually end-irreducible 3-manifold implies the existence of a "nontrivial" proper embedding of a plane into that 3-manifold. We remark that an eventually end-irreducible 3-manifold is essentially a 3-manifold which has an infinite hierarchy. A discussion of proper homotopy and related topics is given in [1].

Notation. All spaces are simplicial complexes and all maps are piecewise linear. We use the notation of Brown and Tucker [1] without change. A 3-manifold is *eventually end-irreducible* at the end $[a]$ if there is an exhausting sequence $\{M_n\}$ of compact 3-dimensional submanifolds of M and a compact subset $C \subset \operatorname{int}(M_1)$ with the following property. (α) If A is the component of $M - M_n$ determined by the end $[a]$ and if F is a component of $\operatorname{Fr}(A)$, then the inclusion map $\pi_1(F) \to \pi_1(M - C)$ induces a monomorphism.

Results. The main result in this paper is the following

THEOREM. *Let M be a 3-manifold and let $[a]$ be an end of M which is eventually end-irreducible. Let f: $(R^2, [\,*\,]) \to (M, [a])$ be a proper map which carries the unique end $[\,*\,]$ of R^2 to the end $[a]$ of M. Assume $\underline{\pi}_1(f)$ is nontrivial. Then there is a proper embedding g: $(R^2, [\,*\,]) \to (M, [a])$ such that $\underline{\pi}_1(g)$ is nontrivial.*

PROOF. We rely heavily on the proof of the loop theorem in [3]. Let C and $\{M_n\}$ be as in the definition of eventually end-irreducible at the end $[a]$. Since $\underline{\pi}_1(f)$ is nontrivial, we may assume that there is a disk $\mathfrak{D}_0 \subset R^2$ so that $f^{-1}(C) \subset \mathfrak{D}_0$ and if λ is an essential loop in $R^2 - \mathfrak{D}_0$, then $f(\lambda)$ is essential in $M - C$.

By a subsequence of the M_n's we may assume that $\mathfrak{D}_0 \subset f^{-1}(M_1)$ and that if $f(R^2)$ meets a component A of $M - M_n$, then A is determined by $[a]$. Next,

Received by the editors November 4, 1974 and, in revised form, March 17, 1975.
AMS (MOS) subject classifications (1970). Primary 55A35; Secondary 57C35.
[1]E. M. Brown and M. S. Brown are partially supported by NSF Grant GP-29076; C. D. Feustel is partially supported by NSF Grant GP-15357.

by an arbitrarily small proper homotopy of f, we assume that $f(R^2) \cap \partial M = \varnothing$; that f is in general position with respect to $\mathrm{Fr}(M_n)$ for every n, and that singularities of f consist of double curves, triple points and nodes. It follows that components of $f^{-1}(M_n)$ are compact submanifolds of R^2 (disks with holes) one of which, say $\mathfrak{D}_n$, contains $\mathfrak{D}_0$.

We claim that without changing $\pi_1(f)$ we can change f to a new map with all the above properties and so that each $\mathfrak{D}_n$ is a disk. (If $\pi_2(a)$ is trivial, the change may be accomplished by a proper homotopy.) To see this consider first $\mathfrak{D}_1$. Exactly one of its boundary components (which we call the outer boundary) bounds a disk in R^2 containing $\mathfrak{D}_1$. Any inner boundary component of $\mathfrak{D}_1$ bounds a disk $\mathfrak{D}$ in R^2 with $f(\mathfrak{D}) \subset (M - C)$ since $f^{-1}(C) \subset \mathfrak{D}_0 \subset \mathfrak{D}_1 \subset \mathfrak{D}_n$. Since $f(\partial \mathfrak{D}) \subset \mathrm{Fr}(M_1)$, property($\alpha$) above implies that we can redefine f on $\mathfrak{D}$ so that $f(\mathfrak{D}) \subset \mathrm{Fr}(M_1)$. We may restore general position by a small proper homotopy. Assume then that $\mathfrak{D}_{n-1}$ is a disk.

Next consider $\mathfrak{D}_n$ for $n > 1$. If $\mathfrak{D}_n$ is not a disk, we can find an inner boundary component of $\mathfrak{D}_n$ which bounds a disk $\mathfrak{D}$ such that $f(\mathfrak{D}) \subset M - C$. Since $\mathrm{Fr}(M_j)$ is incompressible in $M - C$ for all j, we can redefine f on $\mathfrak{D}$ so that $f(\mathfrak{D}) \subset M_n - M_{n-1}$ and so that f still has the general position properties mentioned above. Notice that the above change does not increase the number of boundary components of $\mathfrak{D}_j$ for all j and that $f|\mathfrak{D}_{n-1}$ is unaffected. Thus by induction we may redefine f so that all of the $\mathfrak{D}_n$'s are disks and it should be clear that our new f is still a proper map. Finally since we have not changed f on the outer boundaries of the $\mathfrak{D}_n$'s we have not changed $\pi_1(f)$.

We assume now that $\mathfrak{D}_n$ is a disk for each n, and we note that $f|\partial \mathfrak{D}_n$ is essential in $M - C$. If F_n is the component of $\mathrm{Fr}(M_n)$ containing $f(\partial \mathfrak{D}_n)$, then according to the loop theorem [3], there is an embedded disk E_n in M_n with $E_n \cap \partial M_n = E_n \cap F_n = \partial E_n$. Moreover ∂E_n is not in

$$\ker(\pi_1(F_n) \to \pi_1(M - C)).$$

Recall that in the proof of the loop theorem [3] the disk E_n is constructed as follows: One constructs a tower of 2-sheeted coverings of regular neighborhoods of $f(\mathfrak{D}_n)$. At the top of the tower a disk is selected in the boundary of the regular neighborhood and then brought down the tower by cuts. We observe that E_n will be in general position with respect to $\mathrm{Fr}(M_j)$ for $j < n$ since $f|\mathfrak{D}_n$ is in general position with respect to that surface. Thus we can assume that $E_n \cap \mathrm{Fr}(M_j)$ is a collection of disjoint simple loops. We claim that the possibilities for $E_n \cap \mathrm{Fr}(M_j), j < n$, are essentially determined by the loops $ff^{-1}\mathrm{Fr}(M_j)$ and that there are only finitely many possibilities given $ff^{-1}\mathrm{Fr}(M_j)$. This can be seen by observing that $E_n \cap \mathrm{Fr}(M_j)$ is a collection of loops which are essentially composed of arcs in

$$L = ff^{-1}(\mathrm{Fr}(M_j)) - f(\{x \in f^{-1}(M_j) \colon \{x\} \neq f^{-1}f(x)\})$$

where no arc in L can be used more than twice. (Alternatively, see the addendum to Theorem III.5 in [2].)

We assert next that for each n we can choose an n-tuple $(l_{n,1}, l_{n,2}, \ldots, l_{n,n})$ of simple loops, concentric in the given order on E_n so that $l_{n,i}$ is a component of $E_n \cap \mathrm{Fr}(M_i)$, and so that each $l_{n,i}$ is essential in $M - C$. Clearly we must choose $l_{n,n} = \partial E_n$. Some component of $E_n \cap \mathrm{Fr}(M_{n-1})$ is essential in $M - C$

since $l_{n,n}$ is essential. Choose one and call it $l_{n,n-1}$. Now $l_{n,n-1}$ bounds a subdisk of E_n and some component of the intersection of this subdisk with $\mathrm{Fr}(M_{n-1})$ is essential in $M - C$ since $l_{n,n-1}$ is essential. The truth of the assertion then is demonstrated by a finite induction from the top down.

In the choice of the $l_{n,i}$ we have a certain amount of freedom. Let us pick $l_{n,i}$ on $\mathrm{Fr}(M_i)$ as an innermost loop on E_n which is essential in $M - C$. If $A_{n,i}$ is the subannulus of E_n bounded by $l_{n,i}$ and $l_{n,i+1}$, then $A_{n,i}$ meets ∂M_{i+1} in $l_{n,i+1}$ together with loops which are inessential in $M - C$. These last loops are also inessential on ∂M_{i+1} and hence bound disks there.

It follows that we can define $\overline{E}_n$ such that $\overline{E}_n \cap \mathrm{Fr}(M_j) \subset E_n \cap \mathrm{Fr}(M_j)$ and the disk bounded by $l_{n,j}$ lies within M_j for all $j \leqslant n$.

As noted above there are only a finite number of distinct terms in the sequence $\{l_{n,1}\}$. Let $l_1 = l_{n_i,1}$ for an infinite subsequence $\{n_i\}$ of $\{n\}$. Then choose $l_2 = l_{n_j,2}$ for an infinite subsequence $\{n_j\}$ of $\{n_i\}$. By induction we construct a sequence of simple loops $\{l_k\}$ and a sequence of integers n_k so that for $1 \leqslant j \leqslant k$, $l_j = l_{n_k,j}$. It follows that for each positive integer m, the pair l_m, l_{m+1} bounds the annulus $A_{n_k,m}$ on $\overline{E}_{n_k}$ whenever $k \geqslant m + 1$. Among these annuli, let A_m be one which misses as many of the M_n's as possible. Let A_0 be the subdisk of $\overline{E}_{n_1}$ bounded by l_1.

Now $\bigcup_{m=0}^{\infty} A_m$ is a singular plane in M, which contains the loops $\{l_k\}$ as a concentric proper sequence. Let $g \colon R^2 \to M$ be a map which carries $\{x \mid m \leqslant \|x\| \leqslant m + 1\}$ homeomorphically onto A_m.

We assert that g is a proper map. The only way this can fail is if for some k we have $A_m \cap M_k \neq \emptyset$ for an infinite number of integers m. But if m is one such integer, this means that the annulus on $\overline{E}_{n_i}$ bounded by l_m and l_{m+1} meets $\mathrm{Fr}(M_k)$ in some collection of loops for every $i \geqslant m + 1$ (recall the choice of A_m). But then for i large the number of disjoint simple loops in $\overline{E}_{n_i} \cap \mathrm{Fr}(M_k)$ is as large as we wish. But we pointed out above that this number was bounded by a number independent of n_i. It follows that g is a proper map.

Observe that the loop l_j bounds a subdisk $\mathfrak{D}$ of $\overline{E}_{n_k}$ for $j \leqslant k$ such that $\mathfrak{D} \subset M_j$ and $l_i \subset \mathfrak{D}$ for $i \leqslant j$. Since $\overline{E}_{n_k}$ is an embedded disk it follows that $A_m \cap l_i$ is empty if $i \neq m$ or $m + 1$. We will now describe how to make a sequence of cuts so that we obtain a proper embedding of a plane in M from our map g.

We let $\overline{\mathfrak{D}}_n$ be $\{x \colon \|x\| \leqslant n\}$ and $\overline{A}_n$ be the closure of $\overline{\mathfrak{D}}_{n+1} - \overline{\mathfrak{D}}_n$. Since $g|\overline{\mathfrak{D}}_1$ and $g|\overline{A}_1$ are embeddings the map $g|\overline{\mathfrak{D}}_2$ has no branch points or triple points, furthermore since $g(\overline{\mathfrak{D}}_1) \subset M_1$, we know that the singular set of $g|\overline{\mathfrak{D}}_2$ does not approach the boundary of $\overline{\mathfrak{D}}_2$. Since $l_1 \subset g(\partial \overline{A}_1)$, by cutting and pasting we can obtain a new map $h \colon \overline{\mathfrak{D}}_2 \to M_2$ such that

(1) $h|\overline{\mathfrak{D}}_1 = g|\overline{\mathfrak{D}}_1$,

(2) $h(\partial \overline{\mathfrak{D}}_2) = l_2$,

(3) $h^{-1}\mathrm{Fr}(M_1) \subset g^{-1}\mathrm{Fr}(M_1)$.

We assume that h has been defined on $\overline{\mathfrak{D}}_n$ and proceed inductively by extending h to $\overline{\mathfrak{D}}_{n+1}$. We observe first that $g(\overline{A}_n)$ is an embedding as is $h|\overline{\mathfrak{D}}_n$ and that $l_1, l_2, \ldots, l_n$ are not in $g(\overline{A}_n)$. We may define a singular map h_1 onto $\overline{\mathfrak{D}}_{n+1}$ by

(1) $h_1|\overline{\mathfrak{D}}_n = h$,

(2) $h_1|\overline{A}_n = g|\overline{A}_n$.

Now the singular set of h_1 is made up of a collection of simple double loops none of which meets $\bigcup_{i=1}^{n+1} l_i$. Thus after a sequence of cuts we can use h_1 to define h_2 on $\overline{\mathfrak{D}}_{n+1}$ so that

(1) $h_2 | \overline{\mathfrak{D}}_n = h | \mathfrak{D}_n$,

(2) $h_2^{-1}(\mathrm{Fr}(M_n)) \subset g^{-1}(\mathrm{Fr}(M_n))$,

(3) $h_2 \partial (\overline{\mathfrak{D}}_{n+1}) = l_{n+1}$.

We now extend h to $\mathfrak{D}_{n+1}$ by requiring that $h|\overline{A}_n = h_2|\overline{A}_n$. It follows that we may assume that h has been extended to R^2. Now h is proper since $h^{-1}\mathrm{Fr}(M_n)$ has no more components; then $g^{-1}\mathrm{Fr}(M_n)$ and $h\{x| \; \|x\| = n\} = l_n$. Since $h\{x| \; \|x\| = n\} = l_n$, $\pi_1(h)$ is nontrivial and the theorem follows.

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Department of Mathematics, Dartmouth College, Hanover, New Hampshire 03755 (Current address of E. M. Brown and M. S. Brown)

Department of Mathematics, Virginia Polytechnic Institute and State University, Blacksburg, Virginia 24061 (Current address of C. D. Feustel)