PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 55, Number 2, March 1976

SHORTER NOTES

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A REMARK ON THE FIRST NEIGHBOURHOOD RING OF A NOETHERIAN COHEN-MACAULAY LOCAL RING OF DIMENSION ONE

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ABSTRACT. There is an isomorphism between the first neighbourhood ring of a noetherian Cohen-Macaulay local ring A of dimension one and the ring of endomorphisms of a large power of its maximal ideal.

Let A be a noetherian Cohen-Macaulay local ring of dimension one and $\mathfrak{m}$ be its maximal ideal.

An element a of $\mathfrak{m}^t$ is superficial of degree t if, for every large integer n, $\mathfrak{m}^n a = \mathfrak{m}^{n+t}$. The following results are well known: every superficial element is regular; for every large integer t, there exists a superficial element of degree t [4]. The first neighbourhood ring R of A is the subring $\{(x/y) | x \in \mathfrak{m}^t, y$ superficial of degree $t\}$ of the total quotient ring K of A. For every large integer n, the product $R\mathfrak{m}^n = \mathfrak{m}^n$ [2, 12.1]. Let ν be the least such n.

Let $\operatorname{End}_A(\mathfrak{m}^n)$ denote the algebra of A-endomorphisms of $\mathfrak{m}^n$. There is a sequence

$$A \subset \operatorname{End}_A(\mathfrak{m}) \subset \cdots \subset \operatorname{End}_A(\mathfrak{m}^n) \subset \cdots. \tag{1}$$

THEOREM. 1. *The integer* ν *is the least integer* n *such that* $x\mathfrak{m}^n = \mathfrak{m}^{n+t}$ *for every superficial element* x, *where* t *is the degree of* x.

2. *For every integer* $n \geqslant \nu$, *the ring* $\operatorname{End}_A(\mathfrak{m}^n) = \operatorname{End}_A(\mathfrak{m}^\nu)$ *and there exists an isomorphism* F *of* A*-algebras of* $\operatorname{End}_A(\mathfrak{m}^n)$ *onto* R *such that* $F(\operatorname{Hom}_A(\mathfrak{m}^n, \mathfrak{m}^{n+1}))$ *is the ideal* $R\mathfrak{m}$ *of* R.

PROOF. 1. Let x be a superficial element of degree t. Then $\operatorname{length}(\mathfrak{m}^n/x\mathfrak{m}^n) = te$ where e is the multiplicity of A [2, 12.5]. If $x\mathfrak{m}^n = \mathfrak{m}^{n+t}$, then

$$\operatorname{length}(\mathfrak{m}^n/x\mathfrak{m}^n) = \sum_{i=0}^{t-1} \operatorname{length}(\mathfrak{m}^{n+i}/\mathfrak{m}^{n+i+1}) = te.$$

Received by the editors December 13, 1974.

AMS (MOS) subject classifications (1970). Primary 13H10.

Key words and phrases. Local ring, Cohen-Macaulay, first neighbourhood ring, endomorphisms, superficial element.

As length$(\mathfrak{m}^{n+i}/\mathfrak{m}^{n+i+1})$ is less than e, we must have length$(\mathfrak{m}^n/\mathfrak{m}^{n+1}) = e$. Then $n \geqslant \nu$ [2, 12.10].

On the other hand, x is superficial of degree t if and only if $Rx = R\mathfrak{m}^t$. As $R\mathfrak{m}^\nu = \mathfrak{m}^\nu$, $Rx\mathfrak{m}^\nu = R\mathfrak{m}^t\mathfrak{m}^\nu$ and so $x\mathfrak{m}^\nu = \mathfrak{m}^{\nu+t}$.

2. Let t be an integer such that, for every integer $s \geqslant t$, there exists in $\mathfrak{m}^s$ a superficial element of degree s. Let $b \in \mathfrak{m}^t$ be a superficial element of degree t. If k is a large integer, $a = b^k$ is superficial of degree $s = kt$ and $Ra = \mathfrak{m}^s$. Suppose $n \geqslant \nu$. Then $\mathfrak{m}^n a = \mathfrak{m}^{n+s}$ by 1. If c is superficial of degree $n + s$, then $c = ad$ where $d \in \mathfrak{m}^n$ is superficial of degree n.

Define the homomorphism F: $\mathrm{End}_A(\mathfrak{m}^n) \to R$ by $F(\phi) = \phi(d)/d$.

For every $z \in \mathfrak{m}^n$ and $\phi \in \mathrm{End}_A(\mathfrak{m}^n)$, we have $\phi(zd) = z\phi(d) = d\phi(z)$ and so $\phi(z) = (\phi(d)/d)z$. So F is one to one. On the other hand since $Ra = \mathfrak{m}^s$, every $\lambda \in R$ is x/a where $x \in \mathfrak{m}^s$. But $a\mathfrak{m}^n = \mathfrak{m}^{s+n}$; hence for every $z \in \mathfrak{m}^n$, xz belongs to the ideal $a\mathfrak{m}^n$, so λz belongs to $\mathfrak{m}^n$. Define ϕ by $\phi(z) = \lambda z$. Then $F(\phi) = \lambda$ and F is onto.

If $\phi \in \mathrm{Hom}_A(\mathfrak{m}^n, \mathfrak{m}^{n+1})$, then $\phi(d) \in \mathfrak{m}^{n+1} = R\mathfrak{m}^{n+1}$. As d is superficial of degree n, we have $R\mathfrak{m}^n = Rd$ and so $\phi(d) \in Rd\mathfrak{m}$ and $F(\phi) = \phi(d)/d$ belongs to $R\mathfrak{m}$.

Conversely, if $\alpha \in R\mathfrak{m}$, write $\alpha = \sum \lambda_i e_i$ where $\lambda_i \in R$ and $e_i \in \mathfrak{m}$ to see that the element of $\mathrm{End}_A(\mathfrak{m}^n)$ defined by $\phi(z) = \alpha z$ belongs to $\mathrm{Hom}_A(\mathfrak{m}^n, \mathfrak{m}^{n+1})$.

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