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ON GROUPS OF FINITE WEIGHT

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ABSTRACT. A subset S of a group G is said to normally generate G if the smallest normal subgroup of G which contains S is G itself. If α is minimal with the property that there exist a set of cardinality α which normally generates G then G is said to have weight α. It is shown that if G is a group of finite weight and if the lattice of those normal subgroups of G which are contained in the commutator subgroup G' of G satisfies the minimum condition then the weight of G is equal to the weight of G/G'.

Let G be a group and let S be a subset of G. Then the *normal closure*, $\langle S \rangle$, of S in G is defined to be the smallest normal subgroup of G which contains S. If G is a nontrivial group and if the cardinal α is minimal with the property that G contain a subset of cardinality α whose normal closure is G, then following [1] we say that G has weight α and write $w(G) = \alpha$; we define the weight of the trivial group to be one. Questions concerning the weight of a group and especially the relation between the weight of a group and that of its abelianization arise naturally in the study of knot groups [1], [3]. In particular, it is conjectured by González-Acuña in [1] that for every finite group G, $w(G) = 1$ if and only if $w(G/G') = 1$. (An example of Kervaire [2] shows that the analogous conjecture for finitely generated, infinite groups is false.) The purpose of this paper is to prove the following more general result.

THEOREM. *Let G be a group of finite weight and let $\mathfrak{L}$ be the lattice of normal subgroups of G which are contained in the commutator subgroup G' of G. Then if $\mathfrak{L}$ satisfies the minimum condition, $w(G) = w(G/G')$.*

PROOF. Let N be a normal subgroup of G which is contained in G' and is minimal with the property that $w(G/N) = w(G/G')$. We wish to show that $N = (1)$. Set $w = w(G/G')$ and let $\bar{S} = \{\bar{g}_1, \ldots, \bar{g}_w\}$ be a subset of G/N such that $\langle \bar{S} \rangle = G/N$. For $i = 1, 2, \ldots, w$, pick g_i in G which maps onto $\bar{g}_i$ under the natural map from G to G/N, set $S = \{g_1, \ldots, g_w\}$ and let $K = \langle S \rangle \cap N$. Then since $G = N\langle S \rangle$, if $K = N$, then $G = \langle S \rangle$ so that $N = (1)$. Suppose now that $K \neq N$. Then since N/K is isomorphic to $G/\langle S \rangle$, N/K has finite weight from which we may conclude that N has a proper normal subgroup L which contains K such that $w(N/L) = 1$. We note that L is in fact a normal subgroup of G since G/K is the direct product of its subgroups N/K and $\langle S \rangle/K$. We now show that $w(G/L) = w(G/N)$ from which our theorem will follow immediately.

First, we note that G/L is naturally the direct product of its subgroups

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$\langle S \rangle/K$ and N/L and that N/L is contained in $(G/L)'$. From this it follows that N/L is contained in $(N/L)'$ so that N/L is perfect. Pick x in N so that N/L is the normal closure of the image $\bar{x}$ in N/L of x under the natural map from N to N/L; let $\bar{g}_1$ be the image of g_1 in $\langle S \rangle/K$ under the natural map from $\langle S \rangle$ to $\langle S \rangle/K$ and let $\bar{y}$ be any element of N/L. Then $[\bar{x}\bar{g}_1, \bar{y}] = [\bar{x}, \bar{y}]$ in G/L, and thus if we denote the normal closure in N/L of the set of elements $[\bar{x}, \bar{y}]$, $\bar{y}$ in N/L, by $\overline{M}$, then we see that $\overline{M}$ is contained in $\langle \bar{x}\bar{g}_1 \rangle$. However, since $N/L = \langle \bar{x} \rangle$, $(N/L)/\overline{M}$ is cyclic and since N/L is perfect, we conclude that $N/L = \overline{M}$ so that N/L is contained in $\langle \bar{x}\bar{g}_1 \rangle$ and, hence, N is contained in $\langle xg_1 \rangle L$.

Setting $T = \langle xg_1, g_2, \ldots, g_w \rangle$ and noting that x is an element of N we see that $G = \langle S \rangle N = \langle T \rangle N = \langle T \rangle L$. Thus G/L is the normal closure of the image of T in G/L under the natural map from G to G/L and since $w(G/L) \geqslant w(G/N)$, we have shown that $w(G/L) = w(G/N) = w(G/G')$.

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