

ON EXPOSED POINTS OF THE RANGE OF A VECTOR MEASURE. II

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ABSTRACT. If a weakly compact convex set K in a real Banach space X is strongly exposed by a dense set of functionals in X' , it is proved that the functionals which expose K form a residual set in X' . If $\nu: \mathcal{Q} \rightarrow X$ is a measure, it follows that the set of exposing functionals of its range is a residual G_δ in X' . This, in turn, is found to be equivalent to a theorem of B. Walsh on the residuality of functionals $x' \in X'$ for which $x' \circ \nu \equiv \nu$.

If the set of exposed points of $\nu(\mathcal{Q})$ is weakly closed and ν_A is the restriction of ν to any set $A \in \mathcal{Q}$, it is further proved that every exposed point of the range of ν_A is of the form $\nu(A \cap E)$, where $E \in \mathcal{Q}$ and $\nu(E)$ is an exposed point of $\nu(\mathcal{Q})$.

1. Introduction. Throughout assume X to be a real Banach space, X' the dual of X , and ν a measure defined on a σ -algebra \mathcal{Q} of sets with values in X . As proved by Bartle, Dunford and Schwartz [3], there exists a finite positive measure λ on \mathcal{Q} such that $\nu \equiv \lambda$. As a refinement of this theorem, Rybakov [7] proved that there exists a functional $x' \in X'$ such that the signed measure $x' \circ \nu \equiv \nu$. Such functionals have, in turn, been proved by B. Walsh [9] to form a residual G_δ set in X' .

For any set K in X , it may be recalled that a point $x \in K$ is an *exposed point* [5] of K if there exists an $x' \in X'$ such that $x'(x) > x'(y)$ whenever $y \in K$, $y \neq x$. Such a functional x' is, in turn, said to *expose* the set K at x . A point x of K is further called a *strongly exposed point* [5] of K if there exists an $x' \in X'$ for which (i) $x'(x) > x'(y)$ whenever $y \in K$, $y \neq x$, and (ii) for every net (x_n) in K , $x'(x_n) \rightarrow x'(x)$ implies that $x_n \rightarrow x$. The functional x' is then said to *strongly expose* K at x . The set of exposed points of K is denoted by $\exp K$, and, in case K is convex, the set of its extreme points is denoted by $\text{ext } K$.

If the functionals $x' \in X'$ which strongly expose a weakly compact convex set K in X are dense in X' , then the functionals $x' \in X'$ which expose K are proved in Theorem 2 to be residual in X' ; moreover, if each functional that exposes K is strongly exposing, then they form a residual G_δ set in X' . This yields, in particular, the above theorem of B. Walsh.

For every $A \in \mathcal{Q}$, let ν_A denote the restriction of ν to A , viz. $\nu_A(E) = \nu(A \cap E)$, $E \in \mathcal{Q}$. In case of a finite-dimensional nonatomic measure ν , Husain and Tweddle [4] proved that every extreme point of $\nu_A(\mathcal{Q})$ is of the form $\nu(A \cap E)$, where $E \in \mathcal{Q}$ and $\nu(E) \in \text{ext } \nu(\mathcal{Q})$. If $\nu: \mathcal{Q} \rightarrow X$ is a measure such that the exposed points of its range form a weakly closed set, we obtain

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in Theorem 8 an analogous result for exposed points.

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2. Residuality of exposing functionals. Given a weakly compact convex set K in X , we define, for every $x' \in X'$,

$$K_{x'} = \{x \in K: x'(x) = \max x'(K)\},$$

and the map

$$\rho_K: X' \rightarrow R, \quad \rho_K(x') = \text{diam } K_{x'}, \quad x' \in X'.$$

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LEMMA 1. *For any weakly compact convex set K in X , the map ρ_K is continuous at every $x' \in X'$ which strongly exposes K .*

PROOF. Let $x' \in X'$ strongly expose K at x . Then $\rho_K(x') = 0$. Suppose that ρ_K is not continuous at x' . Then there exists $\epsilon > 0$, and a sequence $\{x'_n\}$ in X' converging to x' for which $\rho_K(x'_n) \geq \epsilon$ for each n . Thus there exist, for each n , a_n and b_n in $K_{x'_n}$ such that $\|a_n - b_n\| \geq \epsilon/2$.

Since K is weakly compact, the sequence $\{a_n\}$ has a subnet (a_j) which converges weakly to some point a in K . Similarly, (b_j) has a subnet (b_j) which converges weakly to some $b \in K$. Now the set $K' = \{x'_n; n \geq 1\} \cup \{x'\}$ is norm-compact, and, as K is bounded, the evaluation map from $K \times K'$ to R , given by $(y, y') \rightarrow y'(y)$, $y \in K$, $y' \in K'$, is jointly continuous relative to the weak topology on K and the norm topology on K' . We thus have $x'_j(a_j) \rightarrow x'(a)$. If $\beta = \sup x'(K)$ and $\beta_j = \sup x'_j(K)$ for every j , then obviously $\beta_j \rightarrow \beta$. Since $a_j \in K_{x'_j}$ for each j , $x'_j(a_j) = \beta_j$, and so we have $x'_j(a_j) \rightarrow \beta$. Thus we obtain $x'(a) = \lim x'_j(a_j) = \beta = x'(x)$. Since $(a_j) \rightarrow a$ weakly, we have $x'(a_j) \rightarrow x'(a)$. As x' strongly exposes K at x , we obtain $a = x$ and $\|a_j - x\| \rightarrow 0$.

By a similar argument, one obtains $x'(b) = \beta$, $b = x$ and $\|b_j - x\| \rightarrow 0$, so that $\|a_j - b_j\| \rightarrow 0$, contrary to the choice of the a_n 's and b_n 's. Hence ρ_K is continuous at x' .

REMARK. It is clear from the above proof that ρ_K is upper semicontinuous when K is norm-compact. It would be interesting to investigate weaker hypotheses on K under which ρ_K is upper semicontinuous.

THEOREM 2. *If the set of strongly exposing functionals of a weakly compact convex set K in X is dense in X' , then the set of its exposing functionals is residual in X' .*

In case every exposing functional of K is further strongly exposing, then they form a residual G_δ set in X' .

PROOF. Let C denote the set of points of continuity of ρ_K , and let X'_e and X'_s denote the sets of functionals $x' \in X'$ which expose or strongly expose K respectively. Since X'_s is, by hypothesis, dense in X' , and ρ_K vanishes at every point of X'_s , ρ_K is zero at every point where it is continuous. But then x' exposes

K whenever $\rho_K(x') = 0$. We thus have, with the help of Lemma 1, $X'_s \subset C \subset X'_e$.

Since the points of continuity of any real valued function form a G_δ set, X'_e contains the dense G_δ set C , and so X'_e is residual in X' . In case $X'_e \subset X'_s$, we have $X'_e = C$, whence the second part of the theorem.

In case $\nu: \mathcal{Q} \rightarrow X$ is a measure, the set $K = \overline{\text{co}} \nu(\mathcal{Q})$ is weakly compact by a theorem of Bartle, Dunford and Schwartz [3]. Hence there exists, according to Theorem 4 of Amir and Lindenstrauss [1], at least one $x' \in X'$ which exposes K . For any other $y' \in X'$ it follows from Theorems 2 and 4 of [2] and Rybakov [7] that all but countably many elements in the segment from x' to y' expose K , and so X'_e is dense in X' . But $X'_e \subset X'_s$ as it follows from the proof of Theorem 5 of [2], and so we have

COROLLARY 3. *If ν is a measure with values in X , the exposing functionals of its range form a residual G_δ set in X' .*

According to Theorems 2 and 4 of [2], an $x' \in X'$ exposes K if and only if $x' \circ \nu \equiv \nu$, and so the above corollary yields in turn

COROLLARY 4 (WALSH [9]). *If ν is a measure with values in X , the functionals $x' \in X'$, for which $x' \circ \nu \equiv \nu$, form a residual G_δ set in X' .*

3. Exposed points of the range of a restriction of a measure. If A and B are two weakly compact convex sets in X , in analogy with the definition of $\text{ext}_B A$ in [4], we define $\text{exp}_B A$ to be the set of those exposed points x of A for which there exists some exposed point y of B such that $x + y \in \text{exp}(A + B)$.

PROPOSITION 5. *If A and B are two weakly compact convex sets in X , then $\text{exp}_B A$ is weakly dense in $\text{exp } A$.*

PROOF. By using Theorem 4 of Amir and Lindenstrauss [1] in place of the Krein-Mil'man Theorem in the proof of Theorem 1 of [4], it may be verified that $\text{exp}_B A$ is weakly dense in $\text{ext } A$, and since $\text{exp}_B A \subset \text{exp } A$ and $\text{exp } A \subset \text{ext } A$, as is well known, it follows that $\text{exp}_B A$ is weakly dense in $\text{exp } A$.

On using Corollary 7 of Troyanski [8] instead, the above proposition is found to hold for the set of strongly exposed points as well.

LEMMA 6. *If $\nu: \mathcal{Q} \rightarrow X$ is a measure, then the weak and norm topologies coincide on the set of extreme points of the closed convex hull of $\nu(\mathcal{Q})$.*

PROOF. There exists a finite positive measure λ on \mathcal{Q} such that $\nu \equiv \lambda$ (see [3]). For each $\phi \in L_\infty(\lambda)$, its Bartle-Dunford-Schwartz integral [3] $\int \phi d\nu \in X$. Let $T: L_\infty(\lambda) \rightarrow X$ be defined by $T(\phi) = \int \phi d\nu$, $\phi \in L_\infty(\lambda)$, and let T_ν denote the restriction of T to the set $P = \{\phi \in L_\infty(\lambda): 0 \leq \phi \leq 1 \text{ } \lambda\text{-a.e.}\}$. It follows from the Radon-Nikodým theorem that T is continuous relative to the weak*-topology $\sigma(L_\infty(\lambda), L_1(\lambda))$ on $L_\infty(\lambda)$ and the weak topology on X . According to a theorem of Bartle, Dunford and Schwartz [3], the set $K = \overline{\text{co}} \nu(\mathcal{Q})$ is weakly compact, and so we have $T_\nu(P) = K$ (see, e.g., [2, Lemma 1]).

We need to show that the weak topology on $\text{ext } K$ is finer than the norm topology. Let (x_i) be a net in $\text{ext } K$ converging weakly to an element x in $\text{ext } K$. Then there exist, by Proposition 2 of [2], unique sets E_i , $E \in \mathcal{Q}$, such

that $T_v^{-1}(x_i) = \{\chi_{E_i}\}$ for each i and $T_v^{-1}(x) = \{\chi_E\}$, where χ_A denotes the characteristic function of $A \in \mathcal{Q}$. It may easily be verified that P is weak*-compact, and since the net (χ_{E_i}) is contained in P , there exists a subnet (χ_{E_j}) of (χ_{E_i}) converging to some $\phi \in P$ relative to the weak*-topology. As $T_v: (P, w^*) \rightarrow (K, w)$ is continuous, $(T_v(\chi_{E_j})) \xrightarrow{w} T_v(\phi)$, i.e. $(x_j) \xrightarrow{w} T_v(\phi)$, and since $(x_j) \xrightarrow{w} x$, we have $T_v(\phi) = x = T_v(\chi_E)$, and so $\phi = \chi_E$ (by [2, Proposition 2]). Now the weak*-topology coincides with the $L_1(\lambda)$ -norm topology on the set of characteristic functions, and so $\|\chi_{E_j} - \chi_E\|_1 \rightarrow 0$. On identifying \mathcal{Q} with the above subset of $L_1(\lambda)$, since $\nu: \mathcal{Q} \rightarrow X$ is continuous relative to the $L_1(\lambda)$ -norm topology on \mathcal{Q} and the norm topology on X , we have $\|\nu(E_j) - \nu(E)\| \rightarrow 0$. Thus every weakly convergent net in $\text{ext } K$ has a subnet that converges in the norm. Hence the lemma.

LEMMA 7. *If $\nu: \mathcal{Q} \rightarrow X$ is a measure, then for every $A \in \mathcal{Q}$ we have*

$$\begin{aligned} \{\nu(E \cap A): \nu(E) \in \exp \nu(\mathcal{Q})\} &\subset \exp \nu_A(\mathcal{Q}) \\ &\subset \{\nu(E \cap A): \nu(E) \in \exp \nu(\mathcal{Q})\}^-. \end{aligned}$$

PROOF. Let $K = \overline{\text{co}} \nu(\mathcal{Q})$, $K_1 = \overline{\text{co}} \nu_A(\mathcal{Q})$ and $K_2 = \overline{\text{co}} \nu_{A^c}(\mathcal{Q})$, where A^c denotes the complement of A . According to [2, Theorem 4], we have $\exp \nu(\mathcal{Q}) = \exp K$ and $\exp \nu_A(\mathcal{Q}) = \exp K_1$. It is easy to see that $K = K_1 + K_2$, and that an $x' \in X'$ exposes K at $\nu(E)$ if and only if x' exposes K_1 at $\nu(E \cap A)$ and K_2 at $\nu(E \cap A^c)$, whence we have $\exp_{K_2} K_1 = \{\nu(E \cap A): \nu(E) \in \exp K\}$. The lemma now follows from Proposition 5 and Lemma 6.

THEOREM 8. *If $\nu: \mathcal{Q} \rightarrow X$ is a measure such that $\exp \nu(\mathcal{Q})$ is weakly closed, then for every $A \in \mathcal{Q}$ we have*

$$\exp \nu_A(\mathcal{Q}) = \{\nu(E \cap A): \nu(E) \in \exp \nu(\mathcal{Q})\}.$$

PROOF. According to Lemma 7, it suffices to prove that the set on the right-hand side is norm-closed. Let $\nu(E_n) \in \exp \nu(\mathcal{Q})$ for each n , and assume that the sequence $\{\nu(E_n \cap A)\}$ converges to $x \in X$. As $\chi_{E_n} \in P$ for each n and (P, w^*) is compact, there exists a subnet (χ_{E_m}) of (χ_{E_n}) and some $\phi \in P$ such that $\chi_{E_m} \xrightarrow{w^*} \phi$. Since $T_v: (P, w^*) \rightarrow (K, w)$ is continuous, the net $(T_v(\chi_{E_m})) \xrightarrow{w} T_v(\phi)$, i.e. $(\nu(E_m)) \xrightarrow{w} T_v(\phi)$. As $\nu(E_m) \in \exp \nu(\mathcal{Q})$ for each m and $\exp \nu(\mathcal{Q})$ is weakly closed, we have $T_v(\phi) \in \exp \nu(\mathcal{Q}) (\subset \text{ext } K)$, and so, by Proposition 2 of [2] there exists a unique set $E \in \mathcal{Q}$ such that $\phi = \chi_E$. But then $\chi_{E_m} \xrightarrow{w^*} \chi_E$ and we obtain, as in the proof of Lemma 6, $\|\chi_{E_m} - \chi_E\|_1 \rightarrow 0$, whence it follows that $\|\chi_{E_m \cap A} - \chi_{E \cap A}\|_1 \rightarrow 0$. Since $\nu: \mathcal{Q} \rightarrow X$ is continuous, the net $(\nu(E_m \cap A))$ converges to $\nu(E \cap A)$. By hypothesis, $\|\nu(E_m \cap A) - x\| \rightarrow 0$, and so $x = \nu(E \cap A)$. Since $\nu(E) \in \exp \nu(\mathcal{Q})$, this completes the proof.

REMARK. When $\exp \nu(\mathcal{Q})$ is not closed, the above theorem does not hold, in general, even for a finite-dimensional measure. For, as proved by Rickert [6, Theorem 1], there exists a nonatomic measure ν_1 defined on the σ -algebra of Borel subsets of $A = [0, 1]$ whose range is the closed unit disk in R^2 . Let ν_2 be the Lebesgue measure on the Borel subsets of $B = [1, 2]$ whose range is the

segment from $(0,0)$ to $(1,0)$. Let \mathcal{Q} be the σ -algebra of Borel subsets of $[0,2]$, and define the measure $\nu: \mathcal{Q} \rightarrow R^2$ by $\nu(E) = \nu_1(E \cap A) + \nu_2(E \cap B)$, $E \in \mathcal{Q}$. The range of ν is the convex hull of the two disks with radius unity and centers $(1,0)$ and $(0,0)$. The point $(0,1)$ is an exposed point of $\nu_A(\mathcal{Q})$, and it may easily be verified that $(0,1)$ is not of the form $\nu(E \cap A)$ for any exposed point $\nu(E)$ of the range of ν . We do not know if Theorem 8 is true when $\exp \nu(\mathcal{Q})$ is only norm-closed.

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