PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 55, Number 2, March 1976

PERTURBATIONS OF GROUPS OF AUTOMORPHISMS OF VON NEUMANN ALGEBRAS

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ABSTRACT. We show that if two uniformly continuous representations of a connected abelian group as $*$-automorphisms of a von Neumann algebra are close in norm, then they are conjugate via a single automorphism close to the identity automorphism.

1. In [5] Kadison and Kastler consider von Neumann algebras $\mathfrak{R}$, $\mathfrak{S}$ acting on the same Hilbert space whose unit balls are close in norm and conjecture that such algebras are unitarily equivalent via a unitary close to the identity operator. This has subsequently been resolved in the affirmative for certain algebras by Christensen [1], [2]. Kadison and Ringrose show [6] that if α is a $*$-automorphism of a von Neumann algebra $\mathfrak{R}$ for which $\|\alpha - \iota\| \leqslant k < 2$, then α can be implemented by a unitary U in $\mathfrak{R}$ with $\|U - I\| \leqslant (2 - (4 - k^2)^{1/2})^{1/2}$. ($\iota$ denotes the identity automorphism of $\mathfrak{R}$ and I is the identity operator in $\mathfrak{R}$.)

These results prompted us to pose the following question: Let $\mathfrak{R}$ be a von Neumann algebra, G a group, and suppose α: $g \to \alpha_g$, β: $g \to \beta_g$ are representations of G as $*$-automorphisms of $\mathfrak{R}$. If $\sup_{g \in G}\|\alpha_g - \beta_g\| \leqslant k$, for some sufficiently small k, does there exist a $*$-automorphism γ of $\mathfrak{R}$ for which $\beta_g = \gamma\alpha_g\gamma^{-1}$ for every $g \in G$ and $\|\gamma - \iota\| \leqslant f(k)$, where f is some increasing function of k? We give an affirmative answer when G is a connected abelian topological group and α, β are uniformly continuous representations.

2. For a von Neumann algebra $\mathfrak{R}$, we denote by $\mathfrak{R}_*$ the predual of $\mathfrak{R}$, i.e. the space of all ultraweakly continuous linear functionals on $\mathfrak{R}$. $\mathfrak{L}(\mathfrak{R})$ will be the space of all bounded linear operators on $\mathfrak{R}$, and for an element γ of $\mathfrak{L}(\mathfrak{R})$, $\mathrm{sp}(\gamma)$ will denote the spectrum of γ in $\mathfrak{L}(\mathfrak{R})$. Recall that if γ is a $*$-automorphism of $\mathfrak{R}$, then $\mathrm{sp}(\gamma) \subseteq \mathbf{T}$, where $\mathbf{T}$ is the set of complex numbers of modulus one. If α: $g \to \alpha_g$ is a representation of a group G as $*$-automorphisms of $\mathfrak{R}$, let

$$\mathfrak{R}^\alpha = \{A \in \mathfrak{R}\colon \alpha_g(A) = A \text{ for every } g \in G\}$$

and let $\mathfrak{Z}^\alpha$ denote the centre of $\mathfrak{R}^\alpha$. We say two representations α: $g \to \alpha_g$, β: $g \to \beta_g$ of G *commute* if $\alpha_g\beta_h = \beta_h\alpha_g$ for all $g, h \in G$. A representation α is *uniformly continuous* if $\|\alpha_g - \iota\| \to 0$ as $g \to e$ where e is the identity element of G. For subalgebras $\mathfrak{A}$, $\mathfrak{B}$ of $\mathfrak{R}$ we define as in [5],

Received by the editors March 4, 1975.
AMS (MOS) subject classifications (1970). Primary 46LXX, 46L10.

$$\|\mathfrak{A} - \mathfrak{B}\| = \sup\{d(A, \mathfrak{B}_1), d(B, \mathfrak{A}_1): A \in \mathfrak{A}_1, B \in \mathfrak{B}_1\},$$

where $\mathfrak{A}_1$, $\mathfrak{B}_1$ are the unit balls of $\mathfrak{A}$, $\mathfrak{B}$ respectively.

3. LEMMA. *Let $\alpha: g \to \alpha_g$, $\beta: g \to \beta_g$ be commuting uniformly continuous representations of a connected abelian topological group G as $*$-automorphisms of a von Neumann algebra $\mathfrak{R}$. If $\|\alpha_g - \beta_g\| < 2$ for every $g \in G$, then $\alpha = \beta$.*

PROOF. For $g \in G$, define $\gamma_g = \beta_g^{-1}\alpha_g$; then since α, β commute and G is abelian, $\gamma: g \to \gamma_g$ is a representation of G as $*$-automorphisms of $\mathfrak{R}$. Moreover,

$$\|\gamma_g - \iota\| = \|\alpha_g - \beta_g\| \leqslant \|\alpha_g - \iota\| + \|\beta_g - \iota\|,$$

so γ is uniformly continuous and $\|\gamma_g - \iota\| < 2$ for every g in G. Hence $\gamma(G)$ is a connected commutative subset of $\mathfrak{L}(\mathfrak{R})$. Let $\mathfrak{A}$ be a maximal commutative subalgebra of $\mathfrak{L}(\mathfrak{R})$ containing $\gamma(G)$ and let Φ denote the set of nonzero multiplicative linear functionals on $\mathfrak{A}$. The maximality of $\mathfrak{A}$ implies $\operatorname{sp}(\gamma_g) = \{\varphi(\gamma_g): \varphi \in \Phi\}$ for every $g \in G$. Now each $\varphi \in \Phi$ is norm continuous so $\varphi(\gamma(G))$ is a connected subgroup of $\mathbf{T}$, and hence either $\varphi(\gamma(G)) = \{1\}$ or $\varphi(\gamma(G)) = \mathbf{T}$. We assert $\varphi(\gamma(G)) = \{1\}$. If not, $\varphi(\gamma(G)) = \mathbf{T}$, so choosing g with $\varphi(\gamma_g) = -1$, we have $2 > \|\gamma_g - \iota\| \geqslant |\varphi(\gamma_g) - 1| = 2$, a contradiction. Therefore $\varphi(\gamma(G)) = \{1\}$ for every $\varphi \in \Phi$. Hence $\operatorname{sp}(\gamma_g) = \{1\}$, so $\gamma_g = \iota$ (see for example [3, Lemma 2.3.9]), and thus $\alpha_g = \beta_g$ for all g in G.

THEOREM. *Let $\mathfrak{R}$ be a von Neumann algebra, G a connected abelian topological group, and suppose $\alpha: g \to \alpha_g$, $\beta: g \to \beta_g$ are uniformly continuous representations of G as $*$-automorphisms of $\mathfrak{R}$. If $\sup_{g \in G}\|\alpha_g - \beta_g\| \leqslant k \leqslant 1/9$, then there exists an inner $*$-automorphism γ of $\mathfrak{R}$ such that $\beta_g = \gamma\alpha_g\gamma^{-1}$ for every $g \in G$, and $\|\gamma - \iota\| \leqslant 2^{7/2}k(1 + (1 - 16k^2)^{1/2})^{-1/2}$.*

PROOF. We show first that $\|\mathfrak{R}^\alpha - \mathfrak{R}^\beta\| \leqslant k$. Since $*$-automorphisms of von Neumann algebras are ultraweakly continuous, $\mathfrak{R}^\alpha$ and $\mathfrak{R}^\beta$ are ultraweakly closed $*$-subalgebras of $\mathfrak{R}$, and because G is abelian, there exists an invariant mean μ on the space of all bounded complex valued functions on G [4, Theorem 1.2.1]. For $A \in (\mathfrak{R}^\alpha)_1$, the function $g \to \varphi(\beta_g(A))$ is bounded for each φ in $\mathfrak{R}_*$, and so we may define B to be the continuous linear functional on $\mathfrak{R}_*$ given by $\varphi \to \int \varphi(\beta_g(A))\,d\mu(g)$. Hence $B \in \mathfrak{R}$, $\|B\| \leqslant 1$, and for $h \in G$, $\varphi \in \mathfrak{R}_*$, Hence $B \in \mathfrak{R}$, $\|B\| \leqslant 1$, and for $h \in G$, $\varphi \in \mathfrak{R}_*$,

$$\langle \beta_h(B), \varphi\rangle = \varphi(\beta_h(B)) = \langle B, \varphi_o \beta_h\rangle = \int \varphi(\beta_{hg}(A))\,d\mu(g)$$

$$= \int \varphi(\beta_g(A))\,d\mu(g), \text{ since } \mu \text{ is invariant, } = \langle B, \varphi\rangle.$$

Therefore $B \in (\mathfrak{R}^\beta)_1$ and for $\varphi \in \mathfrak{R}_*$ with $\|\varphi\| \leqslant 1$,

$$|\varphi(A) - \varphi(B)| = \left|\int \varphi(\alpha_g(A))\,d\mu(g) - \int \varphi(\beta_g(A))\,d\mu(g)\right|$$

$$\leqslant \int |\varphi(\alpha_g(A) - \beta_g(A))|\,d\mu(g) \leqslant k.$$

Thus $\|A - B\| \leqslant k$, and similarly given $B \in (\mathfrak{R}^\beta)_1$, we can find $A \in (\mathfrak{R}^\alpha)_1$

with $\|A - B\| \leqslant k$. This implies $\|\mathfrak{R}^\alpha - \mathfrak{R}^\beta\| \leqslant k$.

Applying [1, Lemma 2.2 and Theorem 3.2] we obtain a unitary W in $\mathfrak{R}$ with $\|W - I\| \leqslant 2^{5/2}k(1 + (1 - 16k^2)^{1/2})^{-1/2}$ such that $W\mathfrak{Z}^\alpha W^* = \mathfrak{Z}^\beta$. By [7, Theorem 1], there exists a unitary representation $g \to U_g$ of G into $\mathfrak{R}$ such that U_g implements α_g and $\|U_g - I\| \to 0$ as $g \to e$. Since G is abelian, we have for $g, h \in G$,

$$\alpha_h(U_g) = U_h U_g U_h^* = U_g,$$

and therefore each U_g belongs to $\mathfrak{R}^\alpha$. Moreover, if $A \in \mathfrak{R}^\alpha$, then $U_g A U_g^* = A$, so $U_g \in \mathfrak{Z}^\alpha$ for every $g \in G$. If θ_g, γ are the automorphisms of $\mathfrak{R}$ implemented by $WU_g W^*$ and W, respectively, then θ: $g \to \theta_g$ is a uniformly continuous representation of G satisfying $\theta_g = \gamma\alpha_g\gamma^{-1}$ for every $g \in G$. Furthermore,

$$\begin{aligned}\|\theta_g - \beta_g\| &\leqslant \|\theta_g - \alpha_g\| + \|\alpha_g - \beta_g\| \leqslant 2\|\gamma - \iota\| + k \\ &\leqslant 4\|W - I\| + k \leqslant 2^{9/2}k(1 + (1 - 16k^2)^{1/2})^{-1/2} + k \\ &< 2, \quad \text{since } k \leqslant 1/9.\end{aligned}$$

Hence by the Lemma, $\theta = \beta$, and therefore $\beta_g = \gamma\alpha_g\gamma^{-1}$ for every g in G, with $\|\gamma - \iota\| \leqslant 2^{7/2}k(1 + (1 - 16k^2)^{1/2})^{-1/2}$.

4. **Acknowledgements.** It is a pleasure to record my thanks to my research supervisor, Professor J. R. Ringrose, for his guidance during the preparation of this paper. I also acknowledge the receipt of a research studentship from the Science Research Council.

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