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A REMARK ON NELSON'S BEST HYPERCONTRACTIVE ESTIMATES

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ABSTRACT. By using a combinatorial estimate we provide a new proof of Nelson's best hypercontractive estimates from L^2 to L^4.

Let G be the differential operator

$$-\frac{1}{2}\frac{d}{dx^2}+x\frac{d}{dx} \quad \text{on} \quad L^2(\mathbf{R}, \pi^{-1/2}e^{-x^2}\,dx).$$

Hypercontractive estimates on e^{-tG} have played a key role in constructive quantum field theory; see e.g. [6]. In [5] Nelson proved the estimate

$$\|e^{-tG}f\|_p \leqslant \|f\|_q \tag{1}$$

if

$$e^{-t} \leqslant \sqrt{(q-1)/(p-1)} \tag{2}$$

where $\|\cdot\|_p$ is the $L^p(\mathbf{R}, \pi^{-1/2}e^{-x^2}\,dx)$ norm. (1) is a "best possible" estimate in the sense that if (2) fails then e^{-tG} is not even bounded from L^p to L^q. Nelson's proof is quite complicated and the beautiful alternate proof of Gross [2] involves some computation. Our goal in this note is to give a simple proof of (1) in case $q = 2$; $p =$ even integer. This is not the first time that hypercontractive estimates have been sharper or easier for this case; see the situation for fermions [3].

Our proof proceeds by a slight strengthening of an argument of Nelson [5] who easily proves (1) with $p = 4$; $q = 2$ if $e^{-t} \leqslant \sqrt{1/4}$. Nelson's argument extends to $p = 2k$, $q = 2$ if $e^{-t} \leqslant \sqrt{1/2k}$ ($k =$ integer). Let $A_k(n)$ be defined as follows. Consider $2kn$ objects broken into $2k$ groups of n objects each. $A_k(n)$ is the number of ways of assigning these $2kn$ objects into kn pairs in such a way that no two objects in the same group are paired with each other. Thus e.g.

$$A_1(n) = n!. \tag{3}$$

Obviously, $A_k(n)$ is dominated by the total number of pairings without any restriction and this is $(2kn)!/(kn)!2^{kn}$. From this one finds that

$$A_k(n) \leqslant (2k)^{kn}(A_1(n))^k. \tag{4}$$

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(4) is the basis of the easy Nelson proof mentioned above. By mimicking Nelson's proof, the best estimates from L^2 to L^{2k} follow from the following combinatorial result which is the main result of this note:

THEOREM 1. $A_k(n) \leqslant (2k-1)^{kn} A_1(n)^k$.

PROOF. We will show that

$$A_{kn} \leqslant [(2k-1)/2k]^{kn}[(2kn)(2kn-2)(2kn-4)\cdots 2]. \tag{5}$$

The last factor in (5) is $2^{kn}(kn)!$. By the multinomial theorem $(kn)! \leqslant k^{kn}(n!)^k$ so (5) implies the estimate of the theorem. Let us give an algorithm for finding all allowed pairings and then estimate the number of choices at each stage. Write $2kn$ objects as $\alpha_1^{(1)}, \dots, \alpha_n^{(1)}$; $\alpha_1^{(2)} \cdots ; \cdots, \alpha_n^{(2k)}$. At each stage choose the group with the most unpaired elements left (if several groups have equal numbers left choose the one with smallest group number j (in $\alpha_i^{(j)}$)). In the group $\alpha^{(j)}$ chosen, pair the $\alpha_i^{(j)}$ with i smallest with some element in some other group. This algorithm will clearly yield each allowed pairing once. After m pairs have been chosen, $2kn - 2m$ elements remain. At least $(2kn-2m)/2k$ of those elements lie in the group with the most unpaired elements so at the $(m+1)$st pairing, at most $[(2k-1)/2k](2kn-2m)$ choices are available. This proves the bound (5). $\square$

We would also like to make a remark about the best possible nature of the hypercontractive bounds. For a semigroup e^{-tG} taking 1 into 1, there is a close connection between G having a gap in its spectrum above zero and e^{-tG} being a *contraction* from L^2 to L^4 for some t. Glimm [1] proved that if G has a gap and if e^{-tG} is bounded from L^2 to L^4 for some t_0, it is a contraction for sufficiently large t. Guerra, Rosen and Simon [4] proved that if e^{-tG} generates a Markov process, then e^{-tG} a contraction from L^2 to L^4 implies a mass gap for G. By "running Glimm's proof backwards", we can sharpen the GRS result:

THEOREM 2. *Let T be a reality preserving bounded operator on $L^2(M; d\mu)$; $\mu(M) = 1$ so that* (a) $T1 = 1$, (b) *T is a contraction from L^2 to L^4. Then, $T^*1 = 1$ and $\|T \restriction \{1\}^\perp\| \leqslant \sqrt{1/3}$.*

PROOF. Let $f = \alpha 1 + g$ with $g \in \{1\}^\perp$, α real and g real valued. Then

$$\|f\|_2^4 = \left(\alpha^4 + 2\alpha^2\|g\|_2^2 + \|g\|_2^4\right)$$

and

$$\|Tf\|_4^4 = \alpha^4 + 4\alpha^3\langle 1, Tg\rangle + 6\alpha^2\|Tg\|_2^2 + O(\alpha).$$

By hypothesis: $\|Tf\|_4 \leqslant \|f\|_2$ so taking α large we have $\langle 1, Tg\rangle \leqslant 0$. This implies that $\langle 1, Tg\rangle = 0$ so that $T^*1 = \alpha 1$. Since $\langle 1, T^*1\rangle = \langle T1, 1\rangle = 1$, $\alpha = 1$. Since $\langle 1, Tg\rangle = 0$, taking α large we have

$$6\|Tg\|_2^2 \leqslant 2\|g\|_2^2 \quad \text{or} \quad \|Tg\|_2 \leqslant \sqrt{1/3}\,\|g\|_2. \quad \square$$

REMARK. Thus, the best possible estimate from L^2 to L^4 implies that G has a gap of size 1. If a better estimate held, the gap would be bigger than 1. Since

G has a gap of precisely one, we have the best possible nature of the estimates.

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Note. After the completion of this manuscript I learned of two new proofs of the full best hypercontractive estimates, one by W. Beckner and the other by H. Brascamp and E. Lieb.

References

1. J. Glimm, *Bosen fields with non-linear self-interaction in two dimensions*, Comm. Math. Phys. **8**(1968), 12–25.

2. L. Gross, *Logarithmic Sobolev inequalities*, Amer. J. Math. (to appear).

3. ———, *Existence and uniqueness of physical ground states*, J. Functional Analysis **10**(1972), 52–109.

4. F. Guerra, L. Rosen and B. Simon, *The* $P(\phi)_2$ *quantum field theory as classical statistical mechanics*, Ann. of Math. **101** (1975), 111–259.

5. E. Nelson, *The free Markoff field*, J. Functional Analysis **12**(1973), 211–227.

6. B. Simon, *The* $P(\phi)_2$ *Euclidean (quantum) field theory*, Princeton Univ. Press, Princeton, N.J., 1974.

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