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COMMUTATIVITY OF ENDOMORPHISM RINGS OF IDEALS. II[1]

S. ALAMELU

ABSTRACT. Let R be a commutative ring. In (1), it was proved that a ring R with noetherian total quotient ring is self-injective if and only if the endomorphism ring of every ideal is commutative. We prove here that if the ring is coherent and is its own total quotient ring, then R is self-injective if and only if $\operatorname{Hom}(I, I) = R$ for every ideal I of R.

In (1), we discussed the commutativity of endomorphism rings of ideals. There, we proved that a ring with noetherian total quotient ring is self-injective if and only if the endomorphism ring of every ideal is commutative. The generalization of this to nonnoetherian total quotient rings seems extremely difficult. We will call an element x of R a stable element if every nonzero homomorphism (x) to R is a multiplication by an element of R. We remark that if every element of R is stable, then Hom (I, I) is commutative for every ideal I of R. The converse is in general not true. But, we have some partial answers in this direction. If the ring is coherent and is its own total quotient ring and if $\operatorname{Hom}(I, I) = R$ for every ideal generated by 2 elements then every element of R is stable. On the other hand we have an example of a noncoherent ring which is its own total quotient ring which has a nonstable element but $\operatorname{Hom}(I, I)$ is commutative for every ideal I of R.

We start with an example of a ring which is not self-injective, but $\operatorname{Hom}(I, I)$ is commutative for every ideal I of R.

EXAMPLE. $R = K[x_1, x_2, \ldots]/(x_1^2, x_2^2, \ldots)$ where x_1, x_2, $\ldots$ are indeterminates. R is the inductive limit of the rings $K[x_1, x_2, \ldots, x_n]/(x_1^2, \ldots, x_n^2)$, $n = 1, 2, \ldots$, and each of them is self-injective. It follows that $\operatorname{Hom}(I, I)$ is commutative for every ideal I. It is easy to see that R is not self-injective.

We note that in this example R has the following properties. Every element of R is stable, $\operatorname{Hom}(I, I) = R$ for every finitely generated ideal I. R is further quasi-local and the zero ideal of R is irreducible. We will presently see that these are not isolated phenomena.

We note first the following general result.

PROPOSITION 1. *If every element of R is stable, then* $\operatorname{Hom}(I, I)$ *is commutative for every ideal I of R.*

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PROOF. It is enough if we prove that for any 2 elements $f, g \in \operatorname{Hom}(I, I)$ and $x \in I$, $fg(x) = gf(x)$. But this follows immediately: for, as x being stable, $f(x)$ and $g(x)$ are multiples of x.

PROPOSITION 2. *Let R be a quasi-local ring (a ring with unique maximal ideal). Suppose* $\operatorname{Hom}(I, I) = R$ *for every ideal I generated by two elements, then* (0) *is irreducible.*

PROOF. Suppose (0) is reducible, then $(0) = Q_1 \cap Q_2$. Take $0 \neq x \in Q_1$, $0 \neq y \in Q_2$. Let F be the homomorphism which takes x to x and y to 0 and $ax + by$ to ax. This is a well-defined homomorphism from (x, y) to (x, y). By hypothesis, F is a multiplication by an element of R. Since R has no nontrivial idempotents and $F^2 = F$, we get a contradiction.

DEFINITION. R is said to be coherent if every finitely generated ideal of R is finitely related.

PROPOSITION 3. *Let R be a coherent quasi-local ring which is its own total quotient ring.*

(1) *Suppose* $\operatorname{Hom}(I, I) = R$ *for every ideal I generated by two elements. Then every element of R is stable and hence* $\operatorname{Hom}(I, I)$ *is commutative for every ideal I of R.*

(2) $\operatorname{Hom}(I, I) = R$ *for every ideal I if and only if R is self-injective.*

PROOF. By Proposition 2, (0) is irreducible. An element a of R is stable if and only if whenever $b \notin (a)$, Ann $(a) \not\subset$ Ann b.

Consider x and y any two elements of R and assume y is not a multiple of x. Consider the relation module K of (x, y). $K = \{(s, t) \mid sx + ty = 0\}$. Since R is coherent, K is finitely generated, generated by (s_i, t_i), $i = 1, 2, \ldots, n$. Since y is not a multiple of x, it follows that $I = (t_1, t_2, \ldots, t_n)$ is a proper ideal.

Now $\operatorname{Ann} I \supset \operatorname{Ann}(t_1) \cap \operatorname{Ann}(t_2) \cap \cdots \cap \operatorname{Ann}(t_n)$ and this is $\neq 0$ since (0) is irreducible and $\operatorname{Ann}(t_i) = 0$ implies t_i is a unit. Hence $\operatorname{Ann} I \cap (x) \neq (0)$. Let $t \in R$ be such that $tx \in \operatorname{Ann} I$. Consider the mapping f: $x \mapsto 0$, $y \mapsto tx$, and extend by linearity. This is well defined, for, if $mx + ny = 0$, then $n \in I$, hence $ntx = 0$. By hypothesis, f: $(x,y) \to (x,y)$ is multiplication by an element of R. I.e., there exists $s \in R$ such that $sy = tx \neq 0$ and $sx = 0$. Thus every element of R is stable.

(2) Let f: $J \to R$ be any homomorphism. Since each element $x \in J$ is mapped into a multiple of x, f is actually a map from J to J. Hence f is a multiplication by an element of R. Hence R is self-injective. The converse is obvious.

The result is also true globally by the following

PROPOSITION 4. *Let R be a coherent ring which is its own total quotient ring.*

(1) *Suppose* $\operatorname{Hom}(I, I) = R$ *for every ideal I generated by two elements. Then every element of R is stable and hence* $\operatorname{Hom}(I, I)$ *is commutative for every ideal I of R.*

(2) $\operatorname{Hom}(I, I) = R$ *for every ideal I, if and only if R is self-injective.*

PROOF. (1) Let x, y be any two elements of R. Let $I = (x, y)$. Since I is finitely related, $\operatorname{Hom}(I, I)_P \approx \operatorname{Hom}(I_P, I_P)$, $\forall P \in \operatorname{Spec} R$. By Proposition 3, x is mapped into a multiple of x at each localisation. Let $f \in \operatorname{Hom}((x), R)$.

For every $P \in \operatorname{Spec} R$, there exists $s \in R - P$ such that $f(x)s = tx$. The ideal generated by s for various primes is R, hence $f(x)$ is a multiplication by an element of R. Hence the element x is stable.

(2) Once again if J is any ideal of R any map $J \to R$ is actually a map $J \to J$ and therefore is a multiplication by an element of R.

The converse is obvious.

If R is noetherian with total quotient ring Q, we have seen that Q is self-injective and therefore each element $x \in Q$ is stable, if $\operatorname{Hom}_R(I, I)$ is commutative for all ideals. Proposition 4 says that a similar result is true for coherent R if we further assume $\operatorname{Hom}_Q(I, I) = Q$ for all ideals I of Q. The question is whether we can relax this condition to saying $\operatorname{Hom}(I, I)$ is commutative. At present, we do not have any example of a coherent ring which is its own total quotient ring, $\operatorname{Hom}_R(I, I)$ is commutative for every ideal I of R, and R has a nonstable element. But it seems that at least coherence is a necessary condition for stableness. Our next example is a noncoherent ring which has a nonstable element, and $\operatorname{Hom}_R(I, I)$ is commutative for each ideal I. The idea of the example is motivated by the following general result.

Suppose $\{A_i\}$, $i \in I$, is a family of rings, each of which has the property that the endomorphism rings of ideals are commutative. Suppose for every i, $\operatorname{Ann} A_i = 0$. Then $\oplus A_i$ also has the property, that the endomorphism rings of ideals are commutative. The proof of this is not essentially different from the case of finite families.

LEMMA. *Let A and B be two rings (not necessarily with identity) such that* $\operatorname{Ann} A = \operatorname{Ann} B = 0$. *Suppose for every ideal I in A or B,* $\operatorname{Hom}(I, I)$ *is commutative. Let $C = A \oplus B$. Then C has the property that* $\operatorname{Hom}(I, I)$ *is commutative for every ideal I in C.*

PROOF. Let I be an ideal in $A \oplus B$. Let P_1 and P_2 be projection on A and B. Let $I_i = P_i I$, $i = 1, 2$. Define a map f_i from I_i to I_i, $i = 1, 2$ as follows. Let $a \in I_1$ and $x \in I_2$. $f_1(a) = P_1 f(a, b)$ and $f_2(x) = P_2 f(y, x)$.

CLAIM. f_i, $i = 1, 2$, are well defined. We show that if (a, b) and $(a, b') \in I$, $P_1 f(a, b) = P_1 f(a, b')$. Let $f(a, b) = (a_1, \alpha)$, and $f(a, b') = (a', \alpha')$. Then for any element $t \in A$, $t(a_1 - a') = 0$, i.e., $a_1 = a'$. Hence f_1 is well defined. Similarly f_2 is also well defined. f_1 and f_2 are clearly homomorphisms since f is a homomorphism and P_i are additive.

To show $\operatorname{Hom}(I, I)$ is commutative, it is enough to prove that for any element $(a, b) \in I$, and any two elements $f, g \in \operatorname{Hom}(I, I)$ $fg(a, b) = gf(a, b)$. Now

$$fg(a, b) = f(g_1(a), g_2(b)) = (f_1 g_1(a), f_2 g_2(b)) = (g_1 f_1(a), g_2 f_2(b)) = gf(a, b).$$

EXAMPLE. Let R be the total quotient ring of the ring

$$K[x, y_1, y_2, \ldots]/(y_1^2, y_2^2, \ldots, xy_1, xy_2, \ldots).$$

Then $\operatorname{Hom}_R(I, I)$ is commutative for every ideal I of R and R has a nonstable element.

PROOF. The maximal ideal R' of R is the direct sum of the rings $A = xk[x]_{(x)}$ and $B = Yk[Y]/(Y^2)$ where $Y = (y_1, y_2, \ldots)$ and $Y^2 = (y_1^2, y_2^2, \ldots)$.

$\operatorname{Ann} A = \operatorname{Ann} B = 0$. Let I be an ideal of R and $f, g \in \operatorname{Hom}(I, I)$. Then f_i and g_i, $i = 1, 2$, defined as in the Lemma are well defined homomorphisms. Since A is an integral domain, $f_1 g_1 = g_1 f_1$. As for f_2 and g_2, we notice that (since f and g are R-homomorphisms) they are in fact endomorphisms of I_2 considered as an ideal in $K[Y]/Y^2$, and hence commute. It is clear that $fg = gf$.

Denote by x the coset of x. The homomorphism $x^2 R \to R$ given by $x^2 \mapsto x$ is not a multiplication by an element. Hence x^2 is a nonstable element.

REMARK. In the ring $k[x,y]/(xy, y^2)$, the ideal (x, y) has noncommutative endomorphism ring. For example, take f: $x \mapsto x$, $y \mapsto 0$ and g: $x \mapsto y$, $y \mapsto 0$. One notices that the set of all zero divisors of the ring is the direct sum of $xk[x]$ and $yk[y]/y^2$ and y annihilates the ring $yk[y]/(y^2)$. Thus, in Lemma, the hypothesis that $\operatorname{Ann} A_i = 0$ is necessary.

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DEPARTMENT OF MATHEMATICS, MADURAI UNIVERSITY, MADURAI 625 021, INDIA

Current address: The Ramanujan Institute for Advanced Studies in Mathematics, Madras University, Madras-600 005, India