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ON PURE STATES OF C^*-SUBALGEBRAS

JOEL ANDERSON

In [1, 2.12.21, p. 58] the following question is raised. Let $\mathfrak{A}$ be a C^*-algebra, $\mathfrak{B}$ a C^*-subalgebra of $\mathfrak{A}$ and q a nonzero positive element of $\mathfrak{A}$. Does there exist a state f on $\mathfrak{A}$ such that the restriction of f to $\mathfrak{B}$ is a pure state and $f(q) > 0$? The question was answered in the negative in [2]. Our purpose in this note is to present another proof of this fact.

PROPOSITION. *Let $\mathfrak{A}$ be a C^*-algebra and let q, a and s be elements of $\mathfrak{A}$ such that q is nonzero and positive, a is selfadjoint, and $q = as - sa$. Let $\mathfrak{B}$ be an abelian C^*-subalgebra of $\mathfrak{A}$ which contains a and the identity e. If f is a state on $\mathfrak{A}$ such that the restriction of f to $\mathfrak{B}$ is a pure state, then $f(q) = 0$.*

PROOF. Since f is a pure state on the commutative C^*-algebra $\mathfrak{B}$, f is multiplicative on $\mathfrak{B}$. Let $f(a) = \lambda$. Then by the Cauchy-Schwartz inequality

$$|f(as) - f(a)f(s)|^2 = |f((a - \lambda e)s)|^2 \leq f\big((a - \lambda e)^2\big)f(s^*s) = 0,$$

and similarly, $f(sa) = f(a)f(s)$. Hence, $f(q) = f(as - sa) = 0$.

If s is the unilateral shift on l^2 ($s(\{\lambda_1, \lambda_2, \ldots\}) = \{0, \lambda_1, \lambda_2, \ldots\}$), then $s^*s - ss^* = q$ is the projection of l^2 onto the space of sequences of the form $\{\lambda_1, 0, 0, \ldots\}$. If $a = s + s^*$, then $q = as - sa$, so q, a and s satisfy the hypotheses of the proposition. Hence, we may take $\mathfrak{A}$ to be the C^*-algebra generated by s and $\mathfrak{B}$ to be the C^*-algebra generated by a and e.

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DEPARTMENT OF MATHEMATICS, CALIFORNIA INSTITUTE OF TECHNOLOGY, PASADENA, CALIFORNIA 91125

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