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CHARACTERIZATIONS OF THE SPHERE BY THE CURVATURE OF THE SECOND FUNDAMENTAL FORM

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ABSTRACT. On an ovaloid S with Gaussian curvature $K(\mathrm{I}) > 0$ in Euclidean three-space E^3 the second fundamental form defines a nondegenerate Riemannian metric with curvature $K(\mathrm{II})$. R. Schneider [7] proved that the spheres in Euclidean space E^{n+1} are the only closed hypersurfaces on which the second fundamental form defines a nondegenerate Riemannian metric of constant curvature. For surfaces in E^3 we give a common generalization of Schneider's theorem and the classical theorem of Liebmann [6] (which states that any ovaloid in E^3 with constant Gaussian curvature is a sphere).

We introduce the following notations. Let $x\colon S^n \to E^{n+1}$ be an imbedding so that $S = x(S^n)$ is an ovaloid in Euclidean $(n+1)$-space.

By appropriate orientation the second fundamental form II defines a Riemannian metric. Let (u^i) be local coordinates and let $\Gamma(\mathrm{I})^k_{ij}$, $K(\mathrm{I})$, $\nabla(\mathrm{I})$, ∇_{I} resp. $\Gamma(\mathrm{II})^k_{ij}$, $K(\mathrm{II})$, $\nabla(\mathrm{II})$, ∇_{II} denote Christoffel symbols, Gaussian curvature, covariant differentiation and the first Beltrami operator with respect to the first fundamental form I respective to the second fundamental form II. Let H be the mean curvature and

$$T^k_{ij} := \Gamma(\mathrm{I})^k_{ij} - \Gamma(\mathrm{II})^k_{ij}.$$

In the following we shall use the second fundamental tensor b_{ij} for "raising and lowering the indices". Then ([7, p. 232]) $T_{ijk} := T^h_{ij} b_{hk}$ is totally symmetric and for $n = 2$ we have

$$K(\mathrm{II}) = H + \tfrac{1}{2} T_{ijk} T^{ijk} - (1/8K^2)\cdot \nabla_{\mathrm{II}} K(\mathrm{I}), \tag{1}$$

which easily implies ([2, p. 7])

$$\begin{aligned} &2H(K(\mathrm{II}) - H)\big(H^2 - K(\mathrm{I})\big) \\ &\qquad = \tfrac{1}{2} K(\mathrm{I})\, \nabla_{\mathrm{II}}\big(H, H^2/K(\mathrm{I})\big) - \tfrac{1}{4} \nabla_{\mathrm{I}}\big(H^2/K(\mathrm{I}), K(\mathrm{I})\big). \end{aligned} \tag{2}$$

The following result is a simple consequence of the Gauss-Bonnet-integralformula in case $n = 2$.

3. LEMMA ([5, Koutroufiotis]). *Let S be an ovaloid in E^3. Then each of the assumptions*

(3a) $K(\mathrm{II}) \geqslant (K(\mathrm{I}))^{1/2}$,

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(3b) $K(\mathrm{II}) \leqslant (K(\mathrm{I}))^{1/2}$
implies $K(\mathrm{II}) = (K(\mathrm{I}))^{1/2}$ *on* S.

For the following result of Koutroufiotis we will give another proof.

4. LEMMA. *Let S be an ovaloid* ($K(\mathrm{I}) > 0$) *in* E^3. *Then* $K(\mathrm{II}) = (K(\mathrm{I}))^{1/2}$ *on S implies that S is a sphere.*

PROOF. Define $g\colon S \to \mathbf{R}$ by $g(q) := H^2(q)/K(\mathrm{I})(q)$, $q \in S$. We have $g(q) \geqslant 1$ for $q \in S$ and $g(q_0) = 1$ if and only if q_0 is an umbilic. Assume S not to be a sphere. Then there exists $\bar{q} \in S$ with $1 < g(\bar{q}) = \max_{q \in S} g(q)$ and $(\partial g/\partial u^i)(\bar{q}) = 0$, so that the right-hand side of (2) vanishes in $\bar{q}$. On the other hand $K(\mathrm{II}) = K(\mathrm{I})^{1/2}$ implies that the left-hand side of (2) is negative in $\bar{q}$, as $(H^2 - K(\mathrm{I}))(\bar{q}) > 0$, which is a contradiction. So $g \equiv 1$ on S and S is a sphere.

5. COROLLARY. *Let S be an ovaloid.*

(5.1) *Each of the assumptions* (3a), (3b) *implies that S is a sphere* [5].

(5.2) $K(\mathrm{II}) \geqslant H$ *implies that S is a sphere* (*cf.* [3, *Problem* (9.6.b), *p.* 224] *for* $n = 2$).

The following lemma is an analogue to a classical theorem of Hilbert [4, Anhang V].

6. LEMMA. *Let S be an ovaloid in E^3. If there exists $q_0 \in S$ where $K(\mathrm{II})$ takes its minimum and $K(\mathrm{I})$ takes its maximum, then S is a sphere.*

PROOF. As $T_{ijk}T^{ijk} \geqslant 0$ and $H \geqslant (K(\mathrm{I}))^{1/2}$, (1) implies

$$K(\mathrm{II}) \geqslant (K(\mathrm{I}))^{1/2} - \frac{1}{8K^2}\,\nabla_{\mathrm{II}}(K(\mathrm{I})).$$

As $\nabla_{\mathrm{II}}(K(\mathrm{I}))(q_0) = 0$ we get, for every $q \in S$,

$$K(\mathrm{II})(q) \geqslant K(\mathrm{II})(q_0) \geqslant (K(\mathrm{I})(q_0))^{1/2} \geqslant (K(\mathrm{I})(q))^{1/2};$$

the assertion follows from (5.1).

THEOREM. *Let S be an ovaloid in E^3. If there exists a function $\Phi\colon \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ which is increasing* (*resp. decreasing*) *in both variables and strictly monotonic in at least one of its variables and if*

$$\Phi(K(\mathrm{I})(q), K(\mathrm{II})(q)) = 0$$

for all $q \in S$, then S is a sphere.

PROOF. Assume Φ to be strictly increasing in the second variable and assume that there exists $q_0 \in S$ such that

$$K(\mathrm{I})(q_0) = \max_{q \in S} K(\mathrm{I})(q)$$

but $K(\mathrm{II})(q_0) > \min_{q \in S} K(\mathrm{II})(q) = K(\mathrm{II})(q_1)$. Then

$$\begin{aligned} 0 = \Phi(K(\mathrm{I})(q_0), K(\mathrm{II})(q_0)) &> \Phi(K(\mathrm{I})(q_0), K(\mathrm{II})(q_1)) \\ &\geqslant \Phi(K(\mathrm{I})(q_1), K(\mathrm{II})(q_1)) = 0, \end{aligned}$$

which is a contradiction. So $K(\mathrm{II})(q_0) = \min_{q \in S} K(\mathrm{II})(q)$ and the assertion follows from Lemma 6.

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