

SEQUENTIAL CONVERGENCE TO INVARIANCE IN $BC(G)$

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ABSTRACT. In this note it is shown that weak and strong convergence to invariance are equivalent for a sequence of probabilities acting on $BC(G)$ of a noncompact locally compact group. This result was known for $G = \mathbb{Z}$. For other generalizations of the bounded sequences on \mathbb{Z} , say $BUC(G)$, the result does not hold.

1. Introduction. Let G be a locally compact group (not necessarily abelian or even amenable). Our group will always be Hausdorff and this together with local compactness implies that G is a normal topological space [3, p. 76]. The sup-norm algebra of bounded continuous real valued functions on G will be denoted by $BC(G)$. For any element y in G we denote the left translation operator by y on $BC(G)$ by $T_y f(x) = f(yx)$.

A mean is a member, m , of the dual space $BC(G)^*$ with $m(1) = 1$ and $f \geq 0$ implying $m(f) \geq 0$. (Thus a mean can be regarded as a regular Borel normalized measure on the Stone-Ćech compactification of G .) A mean m is left invariant if $m(T_y f) = m(f)$ for all y in G and all f in $BC(G)$. We will denote the (possibly empty) set of all left invariant means of Λ . As is suggested by G. G. Lorentz's theory of almost convergent sequences [5] we define

$$\mathcal{L} = \Lambda_{\perp} = \{f \text{ in } BC(G) : m(f) = 0 \text{ for all } m \text{ in } \Lambda\}.$$

If $BC(G)$ fails to be amenable then $\Lambda = \emptyset$ and $\mathcal{L} = BC(G)$.

Let $\{m_n\}$ be a sequence of probabilities (regular Borel normalized measures) on G . We say that $\{m_n\}$ acts on \mathcal{L} if $(m_n, f) \rightarrow 0$ for all f in \mathcal{L} . Following M. M. Day [1] we say that $\{m_n\}$ converges weakly to invariance if $(I - T_y)^* m_n \rightarrow 0$ [ω^*] for all y in G and converges strongly to invariance if $\|(I - T_y)^* m_n\| \rightarrow 0$ for all y in G . The norm here is the norm of the dual space of $BC(G)$. It is clear that for measures on G this norm is the usual total variation norm.

2. Results. Some of the results here hold for any translation invariant subspace on G but we obtain our main result only for $BC(G)$. For this reason the entire discussion is in terms of this space.

THEOREM 1. $\mathcal{L} = \overline{\text{span}} \cup \{(I - T_y)f : y \text{ in } G \text{ and } f \text{ in } BC(G)\}$.

PROOF. It is clear that any function of the form $(I - T_y)f$ is in \mathcal{L} and since \mathcal{L} is a norm closed subspace we have the inclusion of the hull in \mathcal{L} . If the hull

Received by the editors February 26, 1975.

AMS (MOS) subject classifications (1970). Primary 43A07, 43A55, 46E15, 47A35, 60B10.

Key words and phrases. Convergence to invariance, Banach limits, means on groups, almost convergence.

is a proper subset of \mathcal{L} there will be a Hahn-Banach functional ϕ vanishing on the hull with $\phi(h) = 1$ for some h in \mathcal{L} . But then ϕ is a left invariant functional since $(I - T_y)^*\phi = 0$ for all y in G . Thus we may write $\phi = a_1\lambda_1 - a_2\lambda_2$ where λ_1 and λ_2 are in Λ . This implies $\phi(h) = 0$ and this contradiction finishes the proof.

THEOREM 2. *A sequence of probabilities $\{m_n\}$ acts on \mathcal{L} iff $\{m_n\}$ is weakly convergent to invariance. In this case $BC(G)$ is amenable.*

PROOF. Suppose $\{m_n\}$ is weakly convergent to invariance. If f is in \mathcal{L} we can approximate f in norm with a finite linear combination of terms of the form $(I - T_y)g$. It follows that $\overline{\lim}|(m_n, f)| < \epsilon$ for all $\epsilon > 0$ so that $\{m_n\}$ acts on \mathcal{L} . Conversely assume $\{m_n\}$ acts on \mathcal{L} . For f in $BC(G)$,

$$(I - T_y)^*m_n(f) = (m_n, (I - T_y)f) \rightarrow 0$$

as $(I - T_y)f$ is clearly in \mathcal{L} for any f and y . Thus we have ω^* convergence. For amenability—if $\Lambda = \phi$ then $\mathcal{L} = BC(G)$ and we have $(m_n, 1) \rightarrow 0$ which is absurd. We can also give a direct argument by taking λ to be any ω^* cluster point of $\{m_n\}$ and it follows easily that λ is in Λ .

Our main result concerns noncompact groups but we will comment briefly on the compact case. If G is compact it is well known that the Haar measure of G , dx , is the unique (left, right, and two-sided) invariant mean for $BC(G) = C(G)$. Thus \mathcal{L} consists of the continuous functions with Haar integral zero. From this it follows easily that $\{m_n\}$ is weakly convergent to invariance iff $m_n \rightarrow dx$ [ω^*]. It is equally clear that strong convergence to invariance obtains iff the Haar absolutely continuous part of m_n converges to dx in total variation norm.

In the noncompact case we need the elementary fact that the compactly supported continuous functions are in \mathcal{L} . This is easily seen as follows: this class of functions is invariant under each T_y so a member of Λ restricted to this class is a finite invariant measure which must vanish in a noncompact group.

We will consider additional consequences of a group supporting a sequence $\{m_n\}$ which acts on \mathcal{L} . We will call $\{m_n\}$ *asymptotically ergodic* if $m_n(xH) \rightarrow 1$ for no proper closed subgroup H on G .

THEOREM 3. *If $\{m_n\}$ acts on \mathcal{L} then $\{m_n\}$ is asymptotically ergodic.*

PROOF. Suppose for some closed subgroup H we do have $m_n(xH) \rightarrow 1$. Let $u \notin xH$ so $uH \cap xH = \emptyset$. Let $y = xu^{-1}$. Define a function f_0 on xH to be 1 and on $y^{-1}xH$ to be 0. Let f be any bounded continuous extension of f_0 to all of G . (Recall that G is a normal topological space so Tietze's extension theorem is valid in G .) Then $(I - T_y)f$ is in \mathcal{L} so $(m_n, (I - T_y)f) \rightarrow 0$. But from the construction of f we also have $(m_n, (I - T_y)f) \equiv 1$. This contradiction finishes the argument.

It is undoubtedly a well-known folk theorem that a single probability measure m on a locally compact group is supported in an open (hence closed) σ -compact subgroup of G . Interest in completeness, ignorance of a reference, and the fact that part of the argument is essential to the main result leads us to outline the proof. By the regularity of m there is a sequence K_n of compact

sets with $m(K_n) \rightarrow 1$. Since G is locally compact it is clear that a sequence of sets V_n can be found so that

- (a) V_n is an open precompact neighborhood of the identity,
- (b) $V_n = V_n^{-1}$,
- (c) $V_n \cdot V_n \subset V_{n+1}$,
- (d) $K_n \subset V_n$.

Then $H = \cup(V_n)$ is clearly an open (and hence closed!) subgroup of G which supports m . Since $H = \cup(H \cap V_n^-)$ we conclude that m is supported by a σ -compact closed subgroup of G . If we have a sequence of probabilities m_n we can set $m = \sum m_n (1/2)^n$ and we use Theorem 3 to obtain

THEOREM 4. *If $\{m_n\}$ acts on \mathcal{L} then G is σ -compact.*

In the proof of the next theorem Tietze's extension will again play a key role. The novelty will be that the set from which the extension is made is a union of compact sets. We will isolate part of the argument here to show that if a sequence of compact sets $\{K_n\}$ goes to infinity fast enough then $\cup(K_n)$ will be closed. Suppose that $\{V_n\}$ satisfies (a), (b) and (c) above and also

- (e) $G = \cup(V_n)$.

If $K_n \cap V_n = \emptyset$ then $\cup(K_n)$ is closed. For if w is in the closure of this set then w is an interior point of some V_m . Now all but a finite number of the K_n are disjoint from V_m . Since the union of a finite number of compact sets is compact clearly w belongs to $\cup(K_n)$.

Now our main result.

THEOREM 5. *Let G be noncompact. Then the following are equivalent:*

- (a) m_n acts on \mathcal{L} ,
- (b) $(I - T_y)^* m_n \rightarrow 0$ [ω^*] for all y in G ,
- (c) $\|(I - T_y)^* m_n\| \rightarrow 0$ for all y in G .

PROOF. We have shown (a) and (b) equivalent in Theorem 2 and clearly (c) implies (b). (This holds without the noncompactness assumption.) We will obtain (c) from (a) by an indirect argument. So we suppose for some fixed y in G that (c) does not obtain. Then (by passing to a subsequence if necessary) we have

$$\|(I - T_y)^* m_n\| \geq r > 0.$$

We want to inductively select approximate support sets for a subsequence of the measures $\lambda_n = |\sigma_n|$ where $\sigma_n = (I - T_y)^* m_n$. Recall that G is the union of a sequence of open precompact sets $\{V_n\}$ with $V_n \cdot V_n \subset V_{n+1}$. We require an infinite subset N of Z_+ so that for each n in N we have a compact set K_n with the properties:

- (a) $\{K_n\}$ are pairwise disjoint.
- (b) Only a finite number of the sets K_n are contained in any one of the sets V_m .
- (c) $\lambda_n(K_n) \rightarrow r$ for n in N .

There is no problem inductively selecting these sets. Property (b) guarantees that $\cup(K_n; n \in N)$ is closed. Next for each n in N we select a continuous function h_n so that $\|h_n\|_\infty < 1$ and $\sigma_n(h_n) \rightarrow r$. Let g_n be the restriction of h_n to the set K_n . Define g to be the function obtained by taking the union of the

graphs of $\{g_n: n \in N\}$. Property (a) guarantees that g is indeed a function while property (b) implies the g is a continuous function on the closed set $K_\infty = \cup(K_n: n \in N)$. Let g_0 be a Tietze extension of g to all of G so that $\|g_0\|_\infty \leq 1$. Then $(I - T_y)g_0$ is in \mathcal{L} but $\lim\text{-sup}(m_n, (I - T_y)g_0) = r > 0$. This contradiction finishes the argument.

Kerstan and Matthes [4] have observed the fact that

$$\|(I - Tw)^*T_n^*m_n\| = \|(I - Tw)^*m_n\|$$

implies a sequence of probabilities $\{m_n\}$ satisfies (c) of the above theorem if and only if $\{T_n^*m_n\}$ satisfies (c). Thus $\{m_n\}$ acts on \mathcal{L} if and only if $\{T_n^*m_n\}$ acts on \mathcal{L} .

We will say that the sequence $\{m_n\}$ is *uniformly flattened* if for each compactly supported continuous function f $m_n(T_y f) \rightarrow 0$ uniformly for y in G . This clearly will not hold for compact groups.

THEOREM 6. *If $\{m_n\}$ acts on \mathcal{L} and G is noncompact then $\{m_n\}$ is uniformly flattened.*

PROOF. If $\{m_n\}$ acts on \mathcal{L} so does $\{T(y_n)^*m_n\}$. Then for f continuous compactly supported $T(y_n)^*m_n(f) = m_n(T(y_n)f) \rightarrow 0$.

3. Remarks and examples. The main result of this paper, (b) implies (c) of Theorem 5, was proven by G. G. Lorentz in the case $G = Z$ as a characterization of limit methods (strong regularity) which act on the Lorentz almost convergent sequences. Now the space of bounded sequences generalizes to a locally compact group in a natural way as $BC(G)$, $BUC(G)$ (left or two-sided), and $L_\infty(G)$. For our methods and results it is essential that the generalization be taken to be $BC(G)$. We will give one example here which will serve several purposes.

Let $a_n > 0$ be chosen so $\sum a_n = 1$ and pick $\delta(x_i)$ to be unit point masses in \mathbf{R} with x_i rational and dense in $[0, 1]$. Let $\lambda = \sum a_n \delta(x_i)$. We will consider the sequence $m_n = \lambda^n$ obtained from λ by convolution product. First this convolution sequence is (asymptotically) ergodic since the (closed) support $\lambda = [0, 1]$. Next we observe that since λ is atomic all the convolution powers are atomic and also carried by the (nonclosed!) subgroup of rationals. Thus it is clear directly that for y irrational $\|(I - T_y)^*\lambda^n\| = 2$. We also observe that for n fixed $\lambda^n(-n + k, n - k) \rightarrow 0$ uniformly in k . Thus we have a sequence $\{m_n\}$ which is both asymptotically ergodic and uniformly flattened which does not act on \mathcal{L} .

For convolution power sequences on locally compact abelian groups necessary and sufficient conditions for $\|(I - T_y)^*\lambda^n\| \rightarrow 0$ are known. Kerstan and Matthes [4] have shown that this happens if λ is ergodic (supported by no closed coset) and λ^n is asymptotically absolutely continuous with respect to the Haar measure of the group.

This same example shows that the choice of $BC(G)$ in Theorem 5 is critical. We can go through the same sequence of definitions for BUC on \mathbf{R} . Foguel [2] has shown that ergodicity alone for a convolution power sequence gives $(I - T_y)^*\lambda^n(f) \rightarrow 0$ for f in $BUC(G)$. Thus there is a bounded continuous Lorentz function on R so that $\lambda^n(f)$ does not converge to 0 while for

each bounded uniformly continuous Lorentz function g we do have for this example $\lambda^n(g) \rightarrow 0$.

ADDED IN PROOF. A study of sequential convergence of measures in a very general setting is made in [6] by Conway.

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