PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 55, Number 2, March 1976

COMPACT LIE GROUPS WHICH ACT ON EUCLIDEAN SPACE WITHOUT FIXED POINTS

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ABSTRACT. It is shown that a compact Lie group G with identity component G_0 acts without fixed points on euclidean space if and only if G_0 is nonabelian or G/G_0 is not of prime power order, which completes earlier work of P. E. Conner and E. E. Floyd, Conner and D. Montgomery, and W.-C. Hsiang and W.-Y. Hsiang.

In this note we describe completely the class of compact Lie groups which can act without fixed points on euclidean space, thus completing earlier work of P. E. Conner and E. E. Floyd [2], followed by that of Conner and D. Montgomery [3] and W.-C. Hsiang and W.-Y. Hsiang [4, 1.9] who extended Conner and Floyd's original idea.

THEOREM. *Let G be a compact Lie group with identity component G_0. Then G acts without fixed points on some euclidean space if and only if G_0 is nonabelian or G/G_0 does not have prime power order.*

If G_0 is abelian and G/G_0 has prime power order p^k and G acts on $\mathbf{R}^n$, then by P. A. Smith theory the fixed point set $(\mathbf{R}_n)^{G_0}$ is nonempty and $\mathbf{Z}$-acyclic. Then $(\mathbf{R}^n)^{G_0}$ is a G/G_0-space and $(\mathbf{R}^n)^G = [(\mathbf{R}^n)^{G_0}]^{G/G_0}$ is nonempty and $\mathbf{Z}_p$-acyclic, again by Smith theory.

We now begin to consider the converse. The following is a fairly well-known topological analogue of induced representations.

INDUCTION LEMMA. *If G is a compact Lie group, $H \subset G$ is a subgroup of finite index k, and X is an H-space, then there is an action of G on X^k, the product of k copies of X, such that the fixed point set $(X^k)^G$ is the diagonal copy of X^H. Further, if $f\colon X \to X$ is an H-map, then the natural map $f^k\colon X^k \to X^k$ is a G-map.*

We indicate a proof, following a suggestion of G. E. Bredon. See also [6, 1.2].

Notice that, by choosing coset representatives for H in G, X^k can be identified with $\mathrm{Maps}^H(G, X)$, the set of H-maps from G to X, where H acts on G by left translation. Then G acts on $\mathrm{Maps}^H(G, X)$ by $(g\varphi)(g') = \varphi(g'g)$, with fixed point set $[\mathrm{Maps}^H(G, X)]^G \cong X^H$, via the map $\varphi \mapsto \varphi(e)$, where e is the identity element in G. If $f\colon X \to X$ is an H-map, then $f^k\colon X^k \to X^k$ is

Received by the editors January 10, 1975 and, in revised form, May 16, 1975.
AMS (MOS) subject classifications (1970). Primary 57E25, 57E10.

[1] Supported in part by NSF grants GP-33031X and GP-36418X1.

[2] Supported in part by the National Science Foundation and the Alfred P. Sloan Foundation.

identified with f_*: $\text{Maps}^H(G, X) \to \text{Maps}^H(G, X)$, where $f_*(\varphi) = f \circ \varphi$ and is easily seen to be a G-map.

Now, returning to the proof of the Theorem, if $G_0 \subset G$ is nonabelian then, according to [4, 1.9], G_0 acts without fixed points on some euclidean space; hence the Induction Lemma implies that G does also. Similarly, if G/G_0 acts without fixed points on euclidean space, then G does also, via the homomorphism $G \to G/G_0$. (If an effective action is desired simply cross with any effective representation of G.) Thus the proof of the Theorem is reduced to the following result.

PROPOSITION. *If G is a finite group not of prime power order, then G acts without fixed points on some euclidean space.*

The method of Conner and Floyd [2] as refined by J. M. Kister [5] shows that it suffices to find a sphere S on which G acts simplicially and without fixed points and a simplicial G-map f: $S \to S$ of degree 0. For one can then form the infinite mapping telescope T associated to f, a contractible simplicial complex on which G acts without fixed points. One then embeds T in a representation of G. An open invariant regular neighborhood of T finally provides the desired action.

The proof of the Proposition proceeds by induction on the order of G. We may assume, using the Induction Lemma, that every proper subgroup of G has prime power order. Then by an elementary result of group theory [7, 6.5.7] G must be solvable. Therefore we may find a surjection $G \to \mathbf{Z}_p$ and an injection $\mathbf{Z}_q \to G$ where p and q are distinct primes and $\mathbf{Z}_p$ and $\mathbf{Z}_q$ denote the cyclic groups of orders p and q, respectively.

Let D_p and D_q be standard 2-disks with boundary circles S_p and S_q on which $\mathbf{Z}_p$ and $\mathbf{Z}_q$ act by rotations. The surjection $G \to \mathbf{Z}_p$ makes D_p into a G-space with one fixed point and the Induction Lemma shows that the $2k$-disk $(D_q)^k$ is a G-space with one fixed point, where k is the index of $\mathbf{Z}_q$ in G. Let S_q^{2k-1} denote the boundary $(2k - 1)$-sphere, on which G acts without fixed points and consider the join $S = S_p * S_q^{2k-1}$, a $(2k + 1)$-sphere on which G acts without fixed points in the obvious way. We claim that S admits a simplicial G-map of degree 0.

To construct the desired G-map, begin by noticing that there are equivariant maps α: $S_p \to S_p$ and β: $S_q \to S_q$ having degrees $1 + mp$ and $1 + nq$, respectively, where m and n are arbitrary integers. (E.g., $\alpha(z) = z^{1+mp}$, where $\mathbf{Z}_p$ acts on S_p by multiplication by $\exp(2\pi i/p)$.) Let $c\beta$: $D_q \to D_q$ be the obvious conical extension of β to all of D_q.

Now α is also a G-map, as is the map $(c\beta)^k$: $(D_q)^k \to (D_q)^k$. Let γ: $S_q^{2k-1} \to S_q^{2k-1}$ be the restriction of $(c\beta)^k$ to the boundary. Notice that $\text{degree}(\gamma) = (1 + nq)^k$.

In his discussion of the Conner and Floyd construction, Bredon [1, pp. 58–62] shows how to construct, given G-maps such as α and γ, another simplicial G-map $\alpha \,\square\, \gamma$: $S_p * S_q^{2k-1} \to S_p * S_q^{2k-1}$ such that $\text{degree}(\alpha \,\square\, \gamma) = \text{degree}(\alpha) + \text{degree}(\gamma) - 1$. Thus $\text{degree}(\alpha \,\square\, \gamma) = mp + (1 + nq)^k$. Since $(p, q) = 1$ we can choose integers n and l such that $1 + nq = lp$. Then $\text{degree}(\alpha \,\square\, \gamma) = mp + (lp)^k$, where m is still an arbitrary integer. Let $m = -l^k p^{k-1}$. Then $\text{degree}(\alpha \,\square\, \gamma) = 0$ as desired.

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