PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 56, April 1976

EULER CHARACTERISTICS OF COMPLETE INTERSECTIONS

JOHN EWING AND SURESH MOOLGAVKAR[1]

ABSTRACT. We point out that a conjecture of Chen and Ogiue, regarding the Euler characteristic of complete intersections, is a simple consequence of a theorem of Hirzebruch.

Let $F_1, F_2, \ldots, F_r$ be nonsingular hypersurfaces of degrees $a_1, a_2, \ldots, a_r$ in complex projective space $\mathbf{C}P^{n+r}$, and suppose that these hypersurfaces are in general position. The intersection $F_1 \cap F_2 \cap \cdots \cap F_r$ is a nonsingular algebraic manifold denoted by $V_n[a_1, a_2, \ldots, a_r]$. In [1] it was conjectured that the Euler characteristic, $\chi(V_n[a_1, a_2, \ldots, a_r]) = n + 1$ if and only if $a_1a_2 \cdots a_r = 1$ in case n is even; and $\chi(V_n[a_1, a_2, \ldots, a_r]) = n + 1$ if and only if either $a_1a_2 \cdots a_r = 1$ or $a_1a_2 \cdots a_r = 2$ in case n is odd. In this short note we point out that this conjecture is a trivial consequence of the result of Hirzebruch [2].

THEOREM 1 (HIRZEBRUCH).

$$\sum_{n=0}^{\infty} \chi(V_n[a_1, a_2, \ldots, a_r])z^n = \frac{a_1a_2 \cdots a_r}{(1-z)^2} \prod_{i=1}^{r} \frac{1}{1 + (a_i - 1)z}.$$

REMARK. Clearly, $a_1a_2 \cdots a_r | \chi(V_n[a_1, a_2, \ldots, a_r])$.
We note that we have explicit expressions in the 2 cases:

$$\chi(V_n[a]) = n + 2 + \big((1-a)^{n+2} - 1\big)/a,$$
$$\chi(V_n[2,2]) = 2(1 + (-1)^n).$$

LEMMA 2.

$$(-1)^n\chi(V_n[a_1, a_2, \ldots, a_r]) = a_r \sum_{k=0}^{n} (a_r - 1)^{n-k}(-1)^k \chi(V_k[a_1, a_2, \ldots, a_{r-1}]).$$

PROOF. This follows immediately by multiplying power series and Theorem 1. Q.E.D.

LEMMA 3. *If* $a_1a_2 \cdots a_r > 2$ *then* $(-1)^n\chi(V_n[a_1, a_2, \ldots, a_r]) \geqslant 0$.

Received by the editors August 25, 1975.
AMS (MOS) subject classifications (1970). Primary 57D20, 14M10.
Key words and phrases. Euler characteristic, complete intersection manifold, complex projective space.
[1]Partially supported by a fellowship from the Office of Research and Development, Indiana University.

PROOF. Using Lemma 2 inductively, the statement reduces to the fact that

$$(-1)^n\chi(V_n[2,2]) \geqslant 0 \quad \text{and} \quad (-1)^n\chi(V[a]) \geqslant 0 \quad \text{for } a > 2.$$

Q.E.D.

LEMMA 4. *If* $a_1a_2 \cdots a_{r-1} > 2$ *then*

$$(-1)^n\chi(V_n[a_1, a_2, \ldots, a_r]) \geqslant (-1)^n a_r\chi(V_n[a_1, a_2, \ldots, a_{r-1}]).$$

PROOF. By Lemmas 2 and 3, $(-1)^n\chi(V_n[a_1, a_2, \ldots, a_r])$ is given as a sum of positive terms, the last of which is $(-1)^n a_r\chi(V_n[a_1, a_2, \ldots, a_{r-1}])$. Q.E.D.

COROLLARY 5. *If* $\chi(V_n[a_1, a_2, \ldots, a_r]) = n + 1$, *then one of the following two cases obtains*:

(i) n *is even and* $a_1a_2 \cdots a_r = 1$.

(ii) n *is odd and either* $a_1a_2 \cdots a_r = 1$ *or* $a_1a_2 \cdots a_r = 2$.

PROOF. Suppose that n is even and $a_1a_2 \cdots a_r \geqslant 2$. Then by the remark following Theorem 1, $a_i > 2$ for each i. Applying Lemma 4 inductively, we have for some $a > 2$,

$$n + 1 = \chi(V_n[a_1, \ldots, a_r]) \geqslant \chi(V_n[a]) > n + 1,$$

a contradiction.

Suppose that n is odd and $a_1a_2 \cdots a_r > 2$. Then, by Lemma 3,

$$\chi(V_n[a_1, a_2, \ldots, a_r]) \leqslant 0,$$

a contradiction. Q.E.D.

We remark that a similar argument shows that the signature,

$$\tau(V_{2n}[a_1, a_2, \ldots, a_r]) = 1$$

if and only if $a_1a_2, \ldots, a_r = 1$.

REFERENCES

1. B. Y. Chen and K. Ogiue, *Complete intersection manifolds with extremal Euler-Poincaré characteristics*, Proc. Amer. Math. Soc. **50** (1975), 121–126.

2. F. Hirzebruch, *Der Satz Von Riemann-Roch in Faisceau-Theoretischer Formulierung Einige Anwendungen und Offene Fragen*, Proc. Internat. Congress Math. (Amsterdam, 1954), vol. III, Noordhoff, Groningen; North-Holland, Amsterdam, 1956, pp. 457–473. MR **19**, 317.

DEPARTMENT OF MATHEMATICS, INDIANA UNIVERSITY, BLOOMINGTON, INDIANA 47401