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TEST MODULES AND COGENERATORS

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ABSTRACT. If $\operatorname{Hom}_R(A, T) = 0$ implies that $A = 0$ for all R-modules A, then the R-module T is a *test module*. The ring R is said to be a *TC-ring* if every test module is a cogenerator. If S is a simple module over a TC-ring then $\operatorname{End}_R E(S)$ is a local semifir. A commutative ring R is a TC-ring if and only if R_M is a P.I.D. for all maximal ideals M of R.

All rings have an identity element and modules are unitary left modules. For an R-module A, $E(A)$ denotes the injective envelope of A and A^k the k-fold direct sum of A. Let R be a ring and T an R-module. We say that T is a *test module* if $\operatorname{Hom}_R(A, T) = 0$ implies $A = 0$ for all R-modules A. The following result was proved by T. Cheatham and R. Cumbie.

THEOREM A [1, THEOREM 3]. *For a ring R the following are equivalent*:
(i) *every test module is a cogenerator*;
(ii) *for all simple modules S, $E(S)$ is contained in every nonzero factor of $E(S)$*.

A ring satisfying conditions (i) and (ii) in Theorem A is called a *TC-ring*.

Let R be a TC-ring, S a simple R-module and set $E = E(S)$. Some of the properties of E which follow easily from Theorem A are set out in the following two lemmas.

LEMMA 1. *Let A be a nonzero factor module of E. Then the following assertions are true*:

(i) *the socle of A is a direct sum of copies of S and A is an essential extension of its socle*;

(ii) *if the socle of A is isomorphic to S^k for some integer $k > 0$, then $A \approx E^k$*;

(iii) *if $f \in \operatorname{End}_R E$ and $f \neq 0$ then f is onto*;

(iv) *if S' is another simple R-module such that $S \not\approx S'$, then* $\operatorname{Hom}_R(E, E(S')) = 0$.

PROOF. Property (i) was noticed in [1] and is a simple consequence of condition (ii) in Theorem A. Now assume that the socle of A is S^k for some $k > 0$. Since A contains E we have $A = E \oplus A'$ and $A' \neq 0$ if $k > 1$. Since A' is again a factor of E it has, in turn, a summand isomorphic to E if $k > 1$. Continuing this we obtain $A \approx E^k$. Next, let $f \in \operatorname{End}_R E, f \neq 0$. Then $f(E) \subseteq E$ and $f(E)$ contains a copy of E. Since E is indecomposable, $f(E) = E$. Finally (iv) is immediate from (i).

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We remark that in view of (iv) in Lemma 1 the ' cofinitely ' generated modules over a TC-ring split into their ' homogeneous' components. For the details the reader is referred to [2, Theorem 4.26].

LEMMA 2. *Let* $Q = \mathrm{End}_R E, f_1, \ldots, f_k \in Q$ *and assume that the intersection* $\mathrm{Ker}\, f_1 \cap \cdots \cap \mathrm{Ker}\, f_k$ *is irredundant. Then the natural monomorphism*

$$h\colon E/(\mathrm{Ker}\, f_1 \cap \cdots \cap \mathrm{Ker}\, f_k) \to E/\mathrm{Ker}\, f_1 \oplus \cdots \oplus E/\mathrm{Ker}\, f_k$$

is an isomorphism. In particular $\mathrm{Ker}\, f_i + (\bigcap_{j\neq i} \mathrm{Ker}\, f_j) = E$ *for all* $1 \leqslant i \leqslant k$.

PROOF. It is well known that h is an essential monomorphism (e.g. [2, Theorem 4.9]). Also, $E/(\mathrm{Ker}\, f_1 \cap \cdots \cap \mathrm{Ker}\, f_k)$ is injective by (ii) of Lemma 1. Thus h is an isomorphism. The second part of Lemma 2 now follows from the fact that $h(\mathrm{Ker}\, f_i) = \oplus_{j\neq i} E/\mathrm{Ker}\, f_j$ and $h(\bigcap_{j\neq i} \mathrm{Ker}\, f_j) = E/\mathrm{Ker}\, f_i$.

By a *local* ring we mean a ring with a unique maximal (left) ideal.

PROPOSITION 3. *The ring of endomorphisms* Q *of* E *is a local semifir i.e.* Q *is a local domain and every finitely generated* (*left*) *ideal of* Q *is free.*

PROOF. Since E is indecomposable, Q is local and (iii) of Lemma 1 implies that Q is a domain. Let $L = Qf_1 + \cdots + Qf_n, f_1, \ldots, f_n \in Q$ be a left ideal of Q. We may assume that $L \neq 0$ and that $\mathrm{Ker}\, f_1 \cap \cdots \cap \mathrm{Ker}\, f_n = \mathrm{Ker}\, f_1 \cap \cdots \cap \mathrm{Ker}\, f_k$ for some $1 \leqslant k \leqslant n$ with $\mathrm{Ker}\, f_1 \cap \cdots \cap \mathrm{Ker}\, f_k$ irredundant. If $f \in L$ then $\mathrm{Ker}\, f \supseteq \mathrm{Ker}\, f_1 \cap \cdots \cap \mathrm{Ker}\, f_k$ and a standard argument shows the existence of $g_1, \ldots, g_k \in Q$ such that $f = g_1 f_1 + \cdots + g_k f_k$. (See e.g. [2, Theorem 5.15, p. 144].) Hence $L = Qf_1 + \cdots + Qf_k$. We claim L is free on $f_1, \ldots, f_k$. Let $g_1 f_1 + \cdots + g_k f_k = 0, g_1, \ldots, g_k \in Q$ and suppose that $g_i(a) \neq 0$ for some $a \in E$ and $1 \leqslant i \leqslant k$. Put $A = \bigcap_{j\neq i} \mathrm{Ker}\, f_j$. Then $E = \mathrm{Ker}\, f_i + A$ by Lemma 2 and $f_i(A) = E$. Now $a = f_i(b)$ for some $b \in A$ and $0 = (g_1 f_1 + \cdots + g_k f_k)(b) = g_i f_i(b) = g_i(a) \neq 0$. Thus $g_i = 0$ for all $1 \leqslant i \leqslant k$ and L is free.

THEOREM 4. *For a commutative ring* R *the following are equivalent*:

(i) R *is a TC-ring*;

(ii) R_M *is a P.I.D. for all maximal ideals* M *of* R;

(iii) *every nonzero factor module of* $E(R/M)$ *is isomorphic to* $E(R/M)$ *for all maximal ideals* M *of* R.

PROOF. The implication (iii) $\Rightarrow$ (i) is clear and the equivalence of conditions (ii) and (iii) is a special case of the Corollary in [3].

Assume (i) and let M be a maximal ideal of R. Set $E = E(R/M)$. Then E is, in a natural way, an R_M module and it is the R_M-injective envelope of the only simple R_M module $R/M \approx R_M/R_M M$. Then R_M is also a TC-ring by (ii) of Theorem A. In other words we may assume that R is a local ring with maximal ideal M. Then E is a cogenerator and we may assume that R is a subring of $\mathrm{End}_R\, E$. Hence R is a domain by Proposition 3. Let $f, g \in R$. We want to show that either $Rf \supseteq Rg$ or $Rg \supseteq Rf$. We may assume that $fg \neq 0$. If $\mathrm{Ker}\, f \cap \mathrm{Ker}\, g$ were irredundant then $E = \mathrm{Ker}\, f + \mathrm{Ker}\, g$ by Lemma 2. But $\mathrm{Ker}\, fg \supseteq \mathrm{Ker}\, f + \mathrm{Ker}\, g$ and $fg \neq 0$. Therefore either $\mathrm{Ker}\, f \supseteq \mathrm{Ker}\, g$ or $\mathrm{Ker}\, g \supseteq \mathrm{Ker}\, f$. Next, the fact that E is the minimal injective cogenerator implies that $\mathrm{Ann}_R\, \mathrm{Ann}_E\, L = L$ for all ideals of L of R (see [2, p. 148, (5.4.3)]). This, in turn, yields $Rf \supseteq Rg$ or $Rg \supseteq Rf$. Thus R is a valuation domain. Let L be a nonzero ideal of R and $a \in L$, $a \neq 0$. Set $K = Ma$. Since the ideals of R are totally ordered we have $K \subseteq L$, K is irreducible and R/K is an essential extension of $Ra/K \approx R/M$. Therefore $E(R/K) \approx E$ and the natural epimorphism $R/K \to R/L$ extends to a homomorphism $E \to E(R/L)$. This shows that R/L is contained in a nonzero factor of E and the socle of R/L is nonzero by Lemma 1. Since L is irreducible, we obtain that $E(R/L) \approx E$. Accordingly, R/L is Artinian for all nonzero ideals of R by [2, Theorem 3.21]. But this can only happen if R is a field or a rank-one discrete valuation ring. In either case, R is a P.I.D.

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