PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 56, April 1976

GENERATING FUNCTIONS FOR SOME CLASSES OF UNIVALENT FUNCTIONS

ZDZISŁAW LEWANDOWSKI, SANFORD MILLER[1] AND ELIGIUSZ ZŁOTKIEWICZ

ABSTRACT. Let $p(z) = e^{i\beta} + p_1 z + p_2 z^2 + \cdots$ be regular in the unit disc Δ with $|\beta| < \pi/2$, and let $\psi(u, v)$ be a continuous function defined in a domain of $\mathbf{C} \times \mathbf{C}$. With some very simple restrictions on $\psi(u, v)$ the authors prove a lemma that $\operatorname{Re} \psi(p(z), zp'(z)) > 0$ implies $\operatorname{Re} p(z) > 0$. This result is then used to generate subclasses of starlike, spirallike and close-to-convex functions.

1. **Introduction.** In a recent paper [7] it was shown that if $f(z) = z + a_2 z^2 + \cdots$ is regular in the unit disc Δ, with $f(z)f'(z)/z \neq 0$ in Δ, and if α is a real number, then

$$\operatorname{Re}\left[(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(\frac{zf''(z)}{f'(z)} + 1\right)\right] > 0, \quad z \in \Delta,$$
$$\Rightarrow \operatorname{Re}[zf'(z)/f(z)] > 0, \quad z \in \Delta; \tag{1}$$

thus showing that functions $f(z)$ in the class of α-convex functions, $\mathfrak{M}_\alpha$, are in fact starlike.

We can set $p(z) = zf'(z)/f(z)$ and then $p(0) = 1$, $p(z) \neq 0$ and we see that condition (1) is equivalent to

$$\operatorname{Re}\left[p(z) + \alpha\frac{zp'(z)}{p(z)}\right] > 0, \quad z \in \Delta,$$
$$\Rightarrow \operatorname{Re} p(z) > 0, \quad z \in \Delta. \tag{2}$$

All of the inequalities in this paper hold uniformly in the unit disc Δ, and in what follows we shall omit the condition "$z \in \Delta$". Furthermore, if we let $\psi(u, v) = u + \alpha v/u$, then (2) becomes

$$\operatorname{Re} \psi(p(z), zp'(z)) > 0 \Rightarrow \operatorname{Re} p(z) > 0. \tag{3}$$

In this paper we prove (3) for a general class of functions $\psi(u, v)$ and then use this result to generate subclasses of starlike, spirallike and close-to-convex functions.

Received by the editors September 14, 1974.

AMS (MOS) subject classifications (1970). Primary 30A32, 34A40; Secondary 30A04, 30A20.

Key words and phrases. Functions with positive real part, Carathéodory functions, univalent functions, starlike functions, spirallike functions, close-to-convex functions.

[1] This work was carried out while the second author was an I.R.E.X. Scholar in Poland.

2. Definitions and a fundamental lemma.

DEFINITION 2.1. Let $u = u_1 + u_2 i$, $v = v_1 + v_2 i$ and let Ψ be the set of functions $\psi(u, v)$ satisfying:

(a) $\psi(u, v)$ is continuous in a domain D of $\mathbf{C} \times \mathbf{C}$,

(b) $(1, 0) \in D$ and $\operatorname{Re} \psi(1, 0) > 0$,

(c) $\operatorname{Re} \psi(u_2 i, v_1) \leqq 0$ when $(u_2 i, v_1) \in D$ and $v_1 \leqq -\frac{1}{2}(1 + u_2^2)$.

EXAMPLES. It is easy to check that each of the following functions belong to Ψ.

$\psi_1(u, v) = u + \alpha v/u$, α real, with $D = [\mathbf{C} - \{0\}] \times \mathbf{C}$.

$\psi_2(u, v) = u + \alpha v$, $\alpha \geqq 0$, with $D = \mathbf{C} \times \mathbf{C}$.

$\psi_3(u, v) = u - v/u^2$ with $D = [\mathbf{C} - \{0\}] \times \mathbf{C}$.

$\psi_4(u, v) = u^2 + v$ with $D = \mathbf{C} \times \mathbf{C}$.

$\psi_5(u, v) = -\ln(v/u^2 + \frac{1}{2})$ with $D = \{u \| u| > \frac{1}{2}\} \times \{v \| v| < \frac{1}{8}\}$.

The class Ψ is closed with respect to addition, and if $\psi \in \Psi$ then $1/\psi \in \Psi$ for perhaps a different domain, and $\alpha\psi \in \Psi$ for any $\alpha > 0$.

Note that condition (c) of Definition 2.1 can be replaced by $\operatorname{Re} \psi(u_2 i, v_1) \leqq 0$ when $(u_2 i, v_1) \in D$ and $v_1 < 0$. Though some generality is lost in considering the resulting class (for example, ψ_5 is lost) it would be much easier to work with algebraically.

We will need the following generalization of Definition 2.1 for some of our later results.

DEFINITION 2.2. Let $b = e^{i\beta}$ where β is real and $|\beta| < \pi/2$. Let $u = u_1 + u_2 i$, $v = v_1 + v_2 i$ and Ψ_b be the set of functions $\psi(u, v)$ satisfying:

(a) $\psi(u, v)$ is continuous in a domain D of $\mathbf{C} \times \mathbf{C}$,

(b) $(b, 0) \in D$ and $\operatorname{Re} \psi(b, 0) > 0$,

(c) $\operatorname{Re} \psi(u_2 i, v_1) \leqq 0$ when $(u_2 i, v_1) \in D$ and

$$v_1 \leqq -\tfrac{1}{2}(1 - 2u_2 \sin \beta + u_2^2)/\cos \beta.$$

Note that $-\frac{1}{2}(1 - 2u_2 \sin \beta + u_2^2)/\cos \beta < 0$.

From the definitions we see that $\Psi_1 = \Psi$. It is easy to check that $\psi_1, \psi_2, \psi_3 \in \Psi_b$ for any b, while $\psi_4 \in \Psi_b$ for $b = e^{i\beta}$ with $|\beta| < \pi/4$, and $\psi_5 \in \Psi_b$ only for $b = 1$.

DEFINITION 2.3. Let $\psi \in \Psi_b$ with corresponding domain D. We denote by $\mathcal{P}_b(\psi)$ those functions $p(z) = b + p_1 z + p_2 z^2 + \cdots$ that are regular in Δ and satisfy:

(i) $(p(z), zp'(z)) \in D$, and

(ii) $\operatorname{Re} \psi(p(z), zp'(z)) > 0$,

when $z \in \Delta$.

The class $\mathcal{P}_b(\psi)$ is not empty since for any $\psi \in \Psi_b$ it is true that $p(z) = b + p_1 z \in \mathcal{P}_b(\psi)$ for $|p_1|$ sufficiently small (depending on ψ).

We now consider the most important result of this paper.

LEMMA 2.1. *If* $p(z) \in \mathcal{P}_b(\psi)$ *then* $\operatorname{Re} p(z) > 0$.

In other words the theorem states that if $\psi \in \Psi_b$, with corresponding domain D, and if $(p(z), zp'(z)) \in D$ then

$$\operatorname{Re} \psi(p(z), zp'(z)) > 0 \Rightarrow \operatorname{Re} p(z) > 0. \tag{4}$$

PROOF. Since $p(z) = e^{i\beta} + p_1 z + p_2 z^2 + \cdots$ is regular in Δ, if we set

$$p(z) = \frac{1 + w(z)}{1 - w(z)} \cos \beta + i \sin \beta, \tag{5}$$

then $w(0) = 0$, $w(z) \neq 1$ and $w(z)$ is a meromorphic function in Δ. We will show that $|w(z)| < 1$ for $z \in \Delta$ which implies $\operatorname{Re} p(z) > 0$. Suppose that $z_0 = r_0 e^{i\theta_0}$ is a point of Δ such that $\max_{|z| \leqq r_0} |w(z)| = |w(z_0)| = 1$. At such a point, by using a result of I. S. Jack [3, Lemma 1] we have

$$z_0 w'(z_0) = \rho w(z_0), \tag{6}$$

where $\rho \geqq 1$.

Since $|w(z_0)| = 1$ and $w(z_0) \neq 1$ we must have

$$(1 + w(z_0))/(1 - w(z_0)) = Ai, \tag{7}$$

where A is real, and thus from (5) we obtain

$$p(z_0) = [A \cos \beta + \sin \beta]i \equiv ei. \tag{8}$$

Differentiating (5) yields $zp'(z) = 2zw'(z) \cos \beta/(1 - w(z))^2$, and thus by using (6), (7) and (8) we obtain

$$z_0 p'(z_0) = -\frac{\rho}{2}(A^2 + 1)\cos \beta = -\frac{\rho}{2} \frac{1 - 2e \sin \beta + e^2}{\cos \beta} \equiv d.$$

Hence at $z = z_0$ we have $\operatorname{Re} \psi(p(z_0), z_0 p'(z_0)) = \operatorname{Re} \psi(ei, d)$ where e and d are real and $d \leqq -\frac{1}{2}(1 - 2e \sin \beta + e^2)/\cos \beta$. Since $\psi \in \Psi_b$, by (c) of Definition 2.2 we must have $\operatorname{Re} \psi(p(z_0), z_0 p'(z_0)) \leqq 0$, which is a contradiction of the fact that $p(z) \in \mathcal{P}_b(\psi)$. Hence $|w(z)| < 1$ and $\operatorname{Re} p(z) > 0$ for $z \in \Delta$.

Remarks. (i) In the special case $b = 1$, the lemma shows that $\mathcal{P}_1(\psi)$ is a subset of $\mathcal{P}$, the class of Carathéodory functions

(ii) If we apply the lemma to the example ψ_1 we obtain implication (2). By applying the lemma to ψ_2 and ψ_3 we obtain

$$\operatorname{Re}[p(z) + \alpha z p'(z)] > 0, \quad \text{with } \alpha \geqq 0, \quad \Rightarrow \operatorname{Re} p(z) > 0,$$

and

$$\operatorname{Re}[p(z) - zp'(z)/p^2(z)] > 0, \quad \text{with } p(z) \neq 0, \quad \Rightarrow \operatorname{Re} p(z) > 0.$$

We see that each $\psi \in \Psi_b$ can be used to generate a subclass of the set of regular functions with positive real part.

Our final result of this section deals with the relationship between the coefficients of any $p(z) \in \mathcal{P}_b(\psi)$ and its generating function ψ.

Theorem 2.1. *If $p(z) = b + p_1 z + p_2 z^2 + \cdots \in \mathcal{P}_b(\psi)$ and $\psi(u, v)$ is a holomorphic function in its domain D of $\mathbf{C} \times \mathbf{C}$, then*

$$p_1[\psi_1(b, 0) + \psi_2(b, 0)] = q_1, \tag{i}$$

$$\begin{aligned} p_1^2[\psi_{11}(b, 0) + 2\psi_{12}(b, 0) + \psi_{22}(b, 0)] \\ + 2p_2[\psi_1(b, 0) + 2\psi_2(b, 0)] = 2q_2 \end{aligned} \tag{ii}$$

where $|q_1|, |q_2| \leqq 2 \operatorname{Re} \psi(b, 0)$.

PROOF. Since $\psi(p(z), zp'(z))$ is a regular function in Δ, it has a Taylor expansion of the form

$$\psi(p(z), zp'(z)) = q_0 + q_1 z + q_2 z^2 + \cdots \tag{9}$$

valid in Δ. Since $\operatorname{Re} \psi(p(z), zp'(z)) > 0$ we must have $|q_1|, |q_2| \leqq 2 \operatorname{Re} q_0 = 2 \operatorname{Re} \psi(b, 0)$. Comparing coefficients in (9) we obtain (i) and (ii).

This theorem enables us to obtain coefficient-bounds very easily without resorting to tedious series methods. For example, applying the theorem to ψ_5 we quickly obtain $|p_1| \leqslant \ln 2$.

3. **Starlike functions.** Let $f(z) = z + a_2 z^2 + \cdots$ be regular in Δ. If $\operatorname{Re} zf'(z)/f(z) > 0$ for $z \in \Delta$ then $f(z)$ is univalent and is said to be a starlike function. We denote the class of such functions by S^*. In this section we will use our principal lemma to generate subclasses of S^* and to extend some results of S. D. Bernardi and R. J. Libera concerning starlikeness of solutions of certain differential equations.

DEFINITION 3.1. Let $\phi(u, v)$ be any continuous function defined on a domain D of $\mathbf{C} \times \mathbf{C}$. We denote by $\mathcal{S}(\phi)$ those functions $f(z) = z + a_2 z^2 + \cdots$ that are regular in Δ with $f(z)f'(z)/z \neq 0$, such that

(i) $(zf'(z)/f(z), zf''(z)/f'(z) + 1) \in D$ and

(ii) $\operatorname{Re} \phi(zf'(z)/f(z), zf''(z)/f'(z) + 1) > 0$

when $z \in \Delta$.

EXAMPLES. For the following examples which involve multivalued functions we can select an appropriate principal value.

$\phi_1(u, v) = u$ with $D = [\mathbf{C} - \{0\}] \times \mathbf{C}$.

$\phi_2(u, v) = v$ with $D = [\mathbf{C} - \{0\}] \times \mathbf{C}$.

$\phi_3(u, v) = (1 - \alpha)u + \alpha v$, with α real and $D = [\mathbf{C} - \{0\}] \times \mathbf{C}$.

$\phi_4(u, v) = u^{1-\gamma} v^{\gamma}$, with γ real and $D = [\mathbf{C} - \{0\}] \times [\mathbf{C} - \{0\}]$.

$\phi_5(u, v) = uv$ with $D = [\mathbf{C} - \{0\}] \times \mathbf{C}$.

$\phi_6(u, v) = -\ln(u/v - \frac{1}{2})$ with $D = \{u \mid \frac{1}{2} < |u| < \frac{3}{2}\} \times \{v \mid \frac{3}{2} < |v|\}$.

Note that $\mathcal{S}(\phi_1) = S^*$, $\mathcal{S}(\phi_2) = C$, the class of convex functions, $\mathcal{S}(\phi_3) = \mathfrak{M}_\alpha$ and $\mathcal{S}(\phi_4) = \mathfrak{L}_\gamma$, the class of gamma-starlike functions [5].

We now show that by suitably restricting ϕ, $\mathcal{S}(\phi)$ will be a nonempty class of starlike functions.

THEOREM 3.1. *Let $u = u_1 + u_2 i$, $v = v_1 + v_2 i$ and $\phi(u, v)$ a function satisfying*:

(a) *$\phi(u, v)$ is continuous in a domain D of* $[\mathbf{C} - \{0\}] \times \mathbf{C}$,

(b) $(1, 1) \in D$ *and* $\operatorname{Re} \phi(1, 1) > 0$,

(c) $\operatorname{Re} \phi(u_2 i, v_2 i) \leqq 0$ *when* $(u_2 i, v_2 i) \in D$ *and* $v_2/u_2 \geqq \frac{3}{2}[1 + 1/(3u_2^2)]$.

Then $\mathcal{S}(\phi)$ is nonempty and $\mathcal{S}(\phi) \subset S^*$.

PROOF. The set $\mathcal{S}(\phi)$ is nonempty since for any ϕ satisfying (a) and (b) it is true that $f(z) = z + a_2 z^2 \in \mathcal{S}(\phi)$ for $|a_2|$ sufficiently small (depending on ϕ).

If $f(z) \in \mathcal{S}(\phi)$ and we set $p(z) = zf'(z)/f(z)$ for $z \in \Delta$, then $p(z) \neq 0$, $p(z)$ is regular, $p(0) = 1$ and

$$\phi(zf'(z)/f(z), zf''(z)/f'(z) + 1) = \phi(p(z), p(z) + zp'(z)/p(z)).$$

Since $\phi(u, v)$ has domain D in $[\mathbf{C} - \{0\}] \times \mathbf{C}$, if we set $r = r_1 + r_2 i$, $s = s_1$

$+ s_2 i$ and $\psi(r,s) = \phi(r, r + s/r)$ then by (a), (b) and (c), $\psi(r,s)$ is continuous in a domain $D_1 = \{(u, u(v-u)) | (u,v) \in D\}$, Re $\psi(1,0) > 0$ and Re $\psi(r_2 i, s_1) \leqq 0$ when $s_1 \leqq -\frac{1}{2}(1 + r_2^2)$. Hence, by Definition 2.1, $\psi \in \Psi$. Since $(p(z), zp'(z)) \in D_1$ and Re $\psi(p(z), zp'(z)) =$ Re $\phi(zf'/f, zf''/f' + 1) > 0$ when $z \in \Delta$, by Lemma 2.1 with $b = 1$ we must have Re $p(z) > 0$. Hence Re $zf'(z)/f(z) > 0$ and $f(z) \in S^*$.

The theorem shows that each ϕ satisfying (a), (b) and (c) generates a subclass of S^*. It is easy to show that examples $\phi_1, \ldots, \phi_6$ satisfy these conditions. For $\phi = \phi_1, \phi_2, \phi_3$, or ϕ_4 we obtain known subclasses of S^*, but as a new example consider ϕ_5. For $\mathbb{S}(\phi_5)$ we have

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right)\left(\frac{zf''(z)}{f'(z)} + 1\right) > 0 \Rightarrow f(z) \in S^*.$$

Note that if ϕ and θ satisfy (a), (b) and (c) then so do $\phi + \theta$, $u\phi/v$, $-uv\phi$ and $1/\phi$ for perhaps different domains.

The following theorem is the analogue of Theorem 2.1 and provides a very quick method for calculating some coefficient inequalities. It can easily be extended to a_n.

THEOREM 3.2. *If $f(z) = z + a_2 z^2 + \cdots \in \mathbb{S}(\phi)$ and $\phi(u,v)$ is a holomorphic function in its domain, then*

(i) $$a_2[\phi_1 + 2\phi_2] = q_1,$$

(ii) $$a_3[4\phi_1 + 12\phi_2] - a_2^2[2\phi_1 + 8\phi_2 - \phi_{11} - 4\phi_{12} - 4\phi_{22}] = 2q_2$$

where $|q_1|, |q_2| \leqq 2$ Re $\phi(1,1)$ *and all partial derivatives are evaluated at* $(1, 1)$.

We close this section by giving an application of Theorem 3.1 to a problem of S. D. Bernardi [1] and R. J. Libera [6]. In [1] it is shown that if $g \in S^*$ then the solution of the differential equation

(10) $$cf(z) + zf'(z) = (1 + c)g(z)$$

is also in S^*, for $c = 1, 2, 3, \ldots$. We will show that $f(z) \in S^*$ for complex c when Re $c \geqq 0$. (By elementary methods it can be shown that (10) has a regular solution provided that c is not a nonnegative integer.)

Differentiating (10) logarithmically we obtain

(11) $$\frac{zg'(z)}{g(z)} = \frac{zf'(z)}{f(z)} \frac{c + zf''(z)/f'(z) + 1}{c + zf'(z)/f(z)} \equiv \phi\left(\frac{zf'}{f}, \frac{zf''}{f'} + 1\right)$$

where $\phi(u,v) = u(c+v)/(c+u)$. Since $g(z) \in S^*$, from (11) we obtain Re $\phi(zf'/f, zf''/f' + 1) > 0$, when $z \in \Delta$, and hence from Definition 3.1 we see that $f(z) \in \mathbb{S}(\phi)$. It is easy to show that ϕ satisfies conditions (a), (b) and (c) of Theorem 3.2 when Re $c \geqq 0$, and consequently we have $f(z) \in S^*$.

The authors wish to thank Professor P. T. Mocanu for this interesting application.

4. **Spirallike functions.** Let $f(z) = z + a_2 z^2 + \cdots$ be regular in Δ and let β be a real number such that $|\beta| < \pi/2$. If Re$[e^{i\beta} zf'(z)/f(z)] > 0$ for z

$\in \Delta$ then $f(z)$ is univalent [8] and is said to be β-spirallike. We denote the class of such functions by $\check{S}(\beta)$. Note that $\check{S}(0) = S^*$.

DEFINITION 4.1. Let $\omega(u, v)$ be any continuous function defined on a domain D of $\mathbf{C} \times \mathbf{C}$. We denote by $\check{\mathbb{S}}_\beta(\omega)$, $|\beta| < \pi/2$, those functions $f(z) = z + a_2 z^2 + \cdots$ that are regular in Δ with $f(z)f'(z)/z \neq 0$, such that

$$\text{(i)} \qquad (e^{i\beta} zf'(z)/f(z), (e^{i\beta} - 1)zf'(z)/f(z) + zf''(z)/f'(z) + 1) \in D$$

and

$$\text{(ii)} \qquad \operatorname{Re} \omega(e^{i\beta} zf'/f, (e^{i\beta} - 1)zf'/f + zf''/f' + 1) > 0$$

when $z \in \Delta$. Note that $\check{\mathbb{S}}_0(\omega) = \mathbb{S}(\omega)$.

Our main result for generating subclasses of spirallike functions is the following theorem which is easily proved by using Lemma 2.1.

THEOREM 4.1. *Let* $u = u_1 + u_2 i$, $v = v_1 + v_2 i$, $|\beta| < \pi/2$ *and* $\omega(u, v)$ *a function satisfying*:

(a) $\omega(u, v)$ *is continuous in a domain* D *of* $[\mathbf{C} - \{0\}] \times \mathbf{C}$,

(b) $(e^{i\beta}, e^{i\beta}) \in D$ *and* $\operatorname{Re} \omega(e^{i\beta}, e^{i\beta}) > 0$,

(c) $\operatorname{Re} \omega(u_2 i, v_2 i) \leqq 0$ *when* $(u_2 i, v_2 i) \in D$ *and*

$$v_2/u_2 \geqq 1 + (1 - u_2 \sin \beta + u_2^2)/2 \cos \beta.$$

Then $\check{\mathbb{S}}_\beta(\omega)$ *is nonempty and* $\check{\mathbb{S}}_\beta(\omega) \subset \check{S}(\beta)$.

The set $\check{\mathbb{S}}_\beta(\omega)$ is not empty since $z \in \check{\mathbb{S}}_\beta(\omega)$ for any ω satisfying the hypothesis. Note that although some generality is lost, the theorem is still true if we replace the inequality in (c) by the simple inequality $v_2/u_2 > 1$.

It is easy to check that the following functions satisfy (a), (b) and (c).

$\omega_1(u, v) = u + \alpha(v - u)$ with α real and $D = [\mathbf{C} - \{0\}] \times \mathbf{C}$,

$\omega_2(u, v) = u(1 + v - u)$ with $D = [\mathbf{C} - \{0\}] \times \mathbf{C}$.

If we apply the theorem to ω_1 we obtain for α real

$$\operatorname{Re}\left[(e^{i\beta} - \alpha)\frac{zf'}{f} + \alpha\left(\frac{zf''}{f'} + 1\right)\right] > 0 \Rightarrow f \in \check{S}(\beta),$$

a result discussed by Eenigenburg et al. [2].

5. **Close-to-convex functions.** Let $f(z) = z + a_2 z^2 + \cdots$ be regular in Δ. If there is a function $g(z) \in C$, the class of convex functions, and a real number β, $|\beta| < \pi/2$, such that $\operatorname{Re}[e^{i\beta} f'(z)/g'(z)] > 0$ for $z \in \Delta$, then $f(z)$ is univalent [4] and is said to be close-to-convex. The class of close-to-convex functions will be denoted by K. In the special case when $g(z) = z$ and $\beta = 0$ we will denote the class by R.

An immediate application of Lemma 2.1 yields the following theorem.

THEOREM 5.1. *Let* $\psi(u, v) \in \Psi$ *with corresponding domain* D. *If* $f(z) = z + a_2 z^2 + \cdots$ *is regular in* Δ *and satisfies*

(i) $(f'(z), zf''(z)) \in D$ *and*

(ii) $\operatorname{Re} \psi(f'(z), zf''(z)) > 0$

when $z \in \Delta$, *then* $f(z) \in R$.

Applying the theorem to ψ_1 when α is real and $f'(z) \neq 0$ we obtain

$$\operatorname{Re}[f'(z) + \alpha z f''(z)/f'(z)] > 0 \Rightarrow f(z) \in R.$$

Similarly for ψ_2, when $\alpha \geqq 0$ we get

$$\operatorname{Re}[f'(z) + \alpha z f''(z)] > 0 \Rightarrow f(z) \in R.$$

A more general means of generating subclasses of close-to-convex functions comes from the following theorem, which is easily proved by using Lemma 2.1.

THEOREM 5.2. *Let $u = u_1 + u_2 i$, $v = v_1 + v_2 i$ and $\omega(u,v)$ be a function satisfying*:
(a) *$\omega(u,v)$ is continuous in a domain D of* $[\mathbf{C} - \{0\}] \times \mathbf{C}$,
(b) $(e^{i\beta}, 0) \in D$ *and* $\operatorname{Re} \omega(e^{i\beta}, 0) > 0$,
(c) $\operatorname{Re} \omega(u_2 i, v_2 i) \leqq 0$ *when* $(u_2 i, v_2 i) \in D$ *and*

$$u_2 v_2 \geqq (1 - u_2 \sin\beta + u_2^2)/2\cos\beta \quad (> 0).$$

Let $g(z) \in C$. If $f(z) = z + a_2 z^2 + \cdots$ is regular in Δ with $f'(z) \neq 0$, $(e^{i\beta} f'(z)/g'(z), zf''(z)/f'(z) - zg''(z)/g'(z)) \in D$, and

$$\operatorname{Re} \omega(e^{i\beta} f'/g', zf''/f' - zg''/g') > 0$$

when $z \in \Delta$ then $f(z) \in K$.

We can apply the theorem to $\omega(u,v) = u(1 + v)$ and obtain

$$\operatorname{Re}\left[e^{i\beta}\frac{f'}{g'}\left(1 + \frac{zf''}{f'} - \frac{zg''}{g'}\right)\right] > 0 \Rightarrow f \in K.$$

BIBLIOGRAPHY

1. S. D. Bernardi, *Convex and starlike univalent functions*, Trans. Amer. Math. Soc. **135** (1969), 429–446. MR **38** #1243.
2. P. J. Eenigenburg, S. S. Miller, P. T. Mocanu and M. O. Reade, *On a subclass of Bazilevič functions*, Proc. Amer. Math. Soc. **45** (1974), 88–92.
3. I. S. Jack, *Functions starlike and convex of order α*, J. London Math. Soc. (2) **3** (1971), 469–474. MR **43** #7611.
4. W. Kaplan, *Close-to-convex schlicht functions*, Michigan Math. J. **1** (1952), 169–185 (1953). MR **14**, 966.
5. Z. Lewandowski, S. Miller and E. Zlotkiewicz, *Gamma-starlike functions*, Ann. Univ. Mariae Curie-Skłodowska **28** (1974), 32–36.
6. R. J. Libera, *Some classes of regular univalent functions*, Proc. Amer. Math. Soc. **16** (1965), 755–758. MR **31** #2389.
7. S. S. Miller, P. T. Mocanu and M. O. Reade, *Bazilevič functions and generalized convexity*, Rev. Roumaine Math. Pures Appl. **19** (1974), 213–224. MR **49** #3105.
8. L. Špăcek, *Contribution à la théorie des fonctions univalentes*, Časopis. Pěst. Mat. **62** (1933), 12–19.

DEPARTMENT OF APPLIED MATHEMATICS, MARIA CURIE-SKŁODOWSKA UNIVERSITY, 20-031, LUBLIN, POLAND

DEPARTMENT OF MATHEMATICS, STATE UNIVERSITY OF NEW YORK, BROCKPORT, NEW YORK 14420

INSTITUTE OF MATHEMATICS, MARIA CURIE- SKŁODOWSKA UNIVERSITY, 20-031 LUBLIN, POLAND