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POSITIVELY CURVED TOTALLY REAL MINIMAL SUBMANIFOLDS IMMERSED IN A COMPLEX PROJECTIVE SPACE

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ABSTRACT. A sufficient condition for a complete totally real minimal submanifold immersed in a complex projective space to be totally geodesic is given in terms of sectional curvature.

1. **Statement of result.** Let $P_n(C)$ be an n-dimensional complex projective space with the Fubini-Study metric of constant holomorphic sectional curvature c, and let M be an n-dimensional complete totally real minimal submanifold immersed in $P_n(C)$.

The purpose of this paper is to prove

THEOREM. *If the sectional curvature of M is greater than $(n-2)c/4(2n-1)$, then M is totally geodesic.*

The result for $n = 2$ was proved by S. T. Yau [3].

2. **Basic lemmas.** We use the same notation and terminologies as in [1] unless otherwise stated. It was proved in [1] that the second fundamental form of the immersion satisfies

$$
\begin{aligned}
\frac{1}{2}\Delta\|\sigma\|^2 = \|\nabla'\sigma\|^2 &+ \sum_{i,j,k,l,m} (h_{ij}^{m^*} h_{kl}^{m^*} R_{lijk} + h_{ij}^{m^*} h_{il}^{m^*} R_{lkjk}) \\
&+ \frac{1}{2}\sum_{i,j} \operatorname{tr}(A_{i^*}A_{j^*} - A_{j^*}A_{i^*})^2 + \frac{c}{4}\|\sigma\|^2.
\end{aligned} \tag{1}
$$

On the one hand, using the equation of Gauss we obtain

$$
\begin{aligned}
&\sum_{i,j,k,l,m} (h_{ij}^{m^*} h_{kl}^{m^*} R_{lijk} + h_{ij}^{m^*} h_{il}^{m^*} R_{lkjk}) \\
&\qquad = \frac{nc}{4}\|\sigma\|^2 + \frac{1}{2}\sum_{i,j} \operatorname{tr}(A_{i^*}A_{j^*} - A_{j^*}A_{i^*})^2 - \sum_{i,j} (\operatorname{tr} A_{i^*}A_{j^*})^2.
\end{aligned} \tag{2}
$$

On the other hand, Yau's idea in [4] can be applied as follows: For each m, let $h_1^m, \ldots, h_n^m$ be the eigenvalues of A_{m^*}. Then we have

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$$\sum_{i,j,k,l} (h_{ij}^{m^*} h_{kl}^{m^*} R_{lijk} + h_{ij}^{m^*} h_{il}^{m^*} R_{lkjk}) = \sum_{i,k} \{h_i^m h_k^m R_{kiik} + (h_i^m)^2 R_{ikik}\}$$
$$= \frac{1}{2} \sum_{i,k} (h_i^m - h_k^m)^2 R_{ikik}.$$

Therefore if the sectional curvature of M is greater than δ, we have

$$\sum_{i,j,k,l} (h_{ij}^{m^*} h_{kl}^{m^*} R_{lijk} + h_{ij}^{m^*} h_{il}^{m^*} R_{lkjk}) \geqslant \frac{1}{2} \sum_{i,k} (h_i^m - h_k^m)^2 \delta$$
$$= n\delta \sum_i (h_i^m)^2 = n\delta \operatorname{tr} A_{m^*}^2,$$

from which it follows that

$$\sum_{i,j,k,l,m} (h_{ij}^{m^*} h_{kl}^{m^*} R_{lijk} + h_{ij}^{m^*} h_{il}^{m^*} R_{lkjk}) \geqslant n\delta \|\sigma\|^2. \tag{3}$$

From (1), (2) and (3) we have

LEMMA 1. *If the sectional curvature of M is greater than δ, then*

$$\frac{1}{2}\Delta\|\sigma\|^2 \geqslant \|\nabla'\sigma\|^2 + (1 + a)n\delta\|\sigma\|^2 - \frac{na - 1}{4} c\|\sigma\|^2$$
$$+ \frac{1 - a}{2} \sum_{i,j} \operatorname{tr}(A_{i^*}A_{j^*} - A_{j^*}A_{i^*})^2 + a \sum_{i,j} (\operatorname{tr} A_{i^*}A_{j^*})^2$$

for $a \geqslant -1$.

The following lemma is purely algebraic.

LEMMA 2. $n^{-1}\|\sigma\|^4 \leqslant \sum_{i,j} (\operatorname{tr} A_{i^*}A_{j^*})^2 \leqslant \|\sigma\|^4$.

3. **Proof of theorem.** Since the symmetric (n, n)-matrix $(\operatorname{tr} A_{i^*}A_{j^*})$ is covariant for the change of basis, for a suitable choice of basis we may assume that

$$\operatorname{tr} A_{i^*}A_{j^*} = 0 \qquad \textit{for } i \neq j. \tag{4}$$

An algebraic lemma of Chern, doCarmo and Kobayashi (Lemma 1 in [2]) implies that

$$\begin{aligned} \sum_{i,j} \operatorname{tr}(A_{i^*}A_{j^*} - A_{j^*}A_{i^*})^2 &\geqslant -2 \sum_{i\neq j} \operatorname{tr} A_{i^*}^2 \operatorname{tr} A_{j^*}^2 \\ &= -2\|\sigma\|^4 + 2 \sum_i (\operatorname{tr} A_{i^*}^2)^2. \end{aligned} \tag{5}$$

From Lemma 1, (4) and (5), it follows that

$$\frac{1}{2}\Delta\|\sigma\|^2 \geqslant (1 + a)n\delta\|\sigma\|^2 - \frac{na - 1}{4} c\|\sigma\|^2 - (1 - a)\|\sigma\|^4 + \sum_i (\operatorname{tr} A_{i^*}^2)^2$$

for $-1 \leqslant a \leqslant 1$.

This, together with Lemma 2 and (4), implies that

$$\frac{1}{2}\Delta\|\sigma\|^2 \geqslant \left\{(1+a)n\delta - \frac{na-1}{4}c\right\}\|\sigma\|^2 + \left\{\frac{1}{n} - (1-a)\right\}\|\sigma\|^4.$$

In particular, putting $a = 1 - 1/n$, we obtain

$$\frac{1}{2}\Delta\|\sigma\|^2 \geqslant \left\{(2n-1)\delta - \frac{n-2}{4}c\right\}\|\sigma\|^2,$$

from which the theorem follows.

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