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ANALYTIC EXPRESSIONS FOR CONTINUED FRACTIONS OVER A VECTOR SPACE

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ABSTRACT. In previous papers, we considered continued fractions obtained from a type of "geometric reciprocal". In this paper we develop certain "analytical" formulas for such continued fractions and, by using these formulas, obtain results analogous to certain classical theorems for ordinary continued fractions.

1. **Introduction**. Throughout this paper, we suppose that S is a real inner product space and that u is a point of S with unit norm. As in [1] and [2], for each point z of S, we denote the point $2((z, u))u - z$ by $\bar{z}$ and the point $\bar{z}/\|z\|^2$ by $1/z$. (We assume that there is adjoined to S a "point at infinity".)

If S is one of E^1, E^2, E^4, and E^8 and u is the point with unit norm and first coordinate 1, the expression

$$\frac{1}{1/x} + \frac{1}{-x} + \frac{1}{(1/x) - y} \tag{1.1}$$

reduces to the product xyx for real numbers, complex numbers, quaternions, and Cayley numbers, respectively. With this in mind, when $y = u$ we denote the value of (1.1) by x^2. If a and b are points of S and a is not a scalar multiple of u, the symbol ab denotes the value of (1.1) when $y = b$ and x is one of the points whose square is a. (If a is not a scalar multiple of u, there are just two points x–one the negative of the other–such that $x^2 = a$. See (2.1) below.) If a is some scalar times u, then ab denotes that scalar times b.

Suppose that each one of $a_1, a_2, a_3, \ldots, b_0, b_1, b_2, \ldots$ is a point of S. Let $T_0(z)$ denote $b_0 + z$ and, for $n = 1, 2, 3, \ldots,$ let $T_n(z)$ denote $a_n(1/(b_n + z))$. With the convention that x/y denotes $x(1/y)$,

$$T_0T_1T_2 \cdots T_n(z) = b_0 + \frac{a_1}{b_1} + \frac{a_1}{b_2} + \cdots + \frac{a_n}{b_n + z}. \tag{1.2}$$

In this paper, we are concerned with the continued fraction generated by the transformations $T_0, T_1, T_2, \ldots,$ i.e., the continued fraction

$$b_0 + \frac{a_1}{b_1} + \frac{a_2}{b_2} + \frac{a_3}{b_3} + \cdots. \tag{1.3}$$

For $z = 0$, we refer to the value of (1.2) as the nth approximant of (1.3).

If each a_n has value u, then (1.3) is the type of continued fraction considered in [1]. In [2], the transformations used to obtain the continued

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fractions were formed somewhat differently. For a sequence $c_1, c_2, c_3, \ldots,$ we defined a transformation S_n by

$$S_n(z) = \begin{cases} \dfrac{1}{c_n} + \dfrac{1}{-c_n} + \dfrac{1}{(1/c_n) - (1/c_n^2) - z} & \text{if } c_n \neq 0, \\ u & \text{if } c_n = 0. \end{cases}$$

Since $-u + 1/S_n(z) = -u + c_n^2(1/c_n^2 + z) = c_n^2 z$ (see §2 below), $S_n(z)$ can be written as $1/(u + c_n^2 z)$. If we set $b_0 = 0$, $a_1 = u$, $b_n = u$, $n = 1, 2, 3, \ldots,$ and $a_{n+1} = c_n^2$, $n = 1, 2, 3, \ldots,$ then the nth approximant of (1.3) is $S_1 S_2 \cdots S_n(u)$. Thus (1.3) includes the continued fractions of [2].

2. **Properties of the operation**. The following Lemma lists the main properties of the operation defined above. We will have occasion to make use of these properties in the last two sections.

LEMMA. *If each one of x, y, and z is a point of S, then*
(a) $x(y + z) = xy + xz$;
(b) $xu = ux = x$;
(c) *if* $x \neq 0$, $x(1/x) = (1/x)x = u$ *and* $x[(1/x)y] = [x(1/x)]y$;
(d) *if* $x \neq 0$ *and* $y \neq 0$, $1/(xy) = (1/x)(1/y)$;
(e) $\|xy\| = \|x\| \, \|y\|$; *and*
(f) *if c is a real number,* $c(xy) = (cx)y = x(cy)$.

This result is readily established with the aid of the following identity: for every x in S and y in S,

$$x^2 y = 2((x, \bar{y}))x - \|x\|^2 \bar{y}. \tag{2.1}$$

To see that this identity holds true, notice that

$$\|x + \bar{y}\|x\|^2\|^2 = \|x\|^2\big(1 + 2((x, \bar{y})) + \|x\|^2\|y\|^2\big),$$

and hence $1/((1/x) - y)$ is the product of $(x - \bar{y}\|x\|^2)\|x\|^2$ times the reciprocal of the right-hand side of this expression. Hence

$$\frac{1}{-x} + \frac{1}{(1/x) - y}$$
$$= \frac{\big(1 - 2((x, \bar{y})) + \|x\|^2\|y\|^2\big)\big(2((x, y))\bar{x} - \|x\|^2\|y\|^2\bar{x} - \|x\|^2 y\big)}{\|2((x, \bar{y}))x - \|x\|^2\|y\|^2 x - \|x\|^2\bar{y}\|^2}.$$

By using the properties of the inner product, the denominator of the right-hand side of this expression can be written so as to obtain

$$\frac{1}{-x} + \frac{1}{(1/x) - y} = \frac{2((x, \bar{y}))\bar{x} - \|x\|^2\|y\|^2\bar{x} - \|x\|^2 y}{\|x\|^4\|y\|^2}.$$

But $\|2((x, \bar{y}))x - \|x\|^2\bar{y}\| = \|x\|^2\|y\|$. Hence the right-hand side of this expression is $1/[2((x, \bar{y}))x - x^2 y] - 1/x$. From this, (2.1) follows.

In general, we do not have $(y + z)x = yx + zx$, $x(yz) = (xy)z$, and $xy = yx$. To see this, we may simply consider appropriate examples in E^4 with $u = (1, 0, 0, 0)$. In this case, xy may be computed in terms of the quaternion product so that examples are easily obtained.

REMARK. As was indicated, in certain cases x^2y is a product of the form xyx. When S is finite dimensional, the cases mentioned are essentially the only ones. In fact, if we suppose that S is finite dimensional and $\cdot$ is a (not necessarily associative) multiplication on S which is distributive and has the property that if x is in S, y is in S, and c is a real number, then $c(x \cdot y) = (cx) \cdot y = x \cdot (cy)$ and if, for every x in S and y in S, $x^2y = (x \cdot y) \cdot x$, then $(S, \cdot)$ is a division algebra and hence S is either E^1, E^2, E^4, or E^8.

3. **Equivalence transformations and some basic formulas.** In this section we give a result which allows us to alter the form of (1.3). We also give some formulas which are closely analogous to the fundamental recurrence formulas for ordinary continued fractions (see [3, pp. 15–20]). At times, the parenthesis indicating a certain association will be omitted with the understanding that the association is from the right to the left. Thus xyz denotes $x(yz)$, $xyzw$ denotes $x(y(zw))$, etc. As before, x/y denotes $x(1/y)$.

THEOREM 1. *Suppose that for* $q = 1, 2, 3, \ldots$ *and* $p = 1, 2, \ldots, q$, c_{pq} *is a point of* S *distinct from* 0. *Then, for* $n = 1, 2, 3, \ldots,$ $T_0T_1T_2 \cdots T_n(0)$*–the nth approximant of* (1.3)*–is equal to*

$$b_0 + \frac{a_1c_{11}}{c_{11}b_1} + \cdots + \frac{c_{n-1,n-1} \cdots c_{1,n-1}a_nc_{1,n} \cdots c_{n,n}}{c_{n,n} \cdots c_{1,n}b_n}. \tag{3.1}$$

By using part (d) of the Lemma, this result may be established by a simple induction argument.

If no a_k is 0, for $q = 1, 2, 3, \ldots,$ $c_{1,q} = 1/a_q$, and, for $q = 2, 3, 4 \ldots$ and $p = 2, 3, \ldots, q$, $c_{pq} = 1/c_{p-1,q-1}$, then (3.1) is equal to

$$b_0 + \frac{1}{b_1'} + \frac{1}{b_2'} + \cdots + \frac{1}{b_n'}$$

where $b_k' = c_{kk} \cdots c_{1,k}b_k$, $k = 1, 2, 3, \ldots.$ If no $b_k = 0$, for $q = 1, 2, 3, \ldots,$ $c_{qq} = 1/b_q$, and for $p \neq q$, $c_{pq} = u$, then (3.1) is equal to

$$b_0 + \frac{a_1'}{u} + \frac{a_2'}{u} + \cdots + \frac{a_n'}{u}$$

where $a_1' = a_1c_{11}$ and $a_k' = c_{k-1,k-1}a_kc_{k,k}$, $k = 2, 3, 4, \ldots.$ Thus, if no a_k is 0 and, for $k = 1, 2, 3, \ldots, b_{2k}' = a_1(1/a_2) \cdots (1/a_{2k})b_{2k}$ and $b_{2k-1}' = (1/a_1)\ a_2 \cdots (1/a_{2k-1})b_{2k-1}$, (1.3) is *equivalent* to

$$b_0 + \frac{1}{b_1'} + \frac{1}{b_2'} + \frac{1}{b_3'} + \cdots, \tag{3.2}$$

while if no b_k is 0 and, for $k = 1, 2, 3, \ldots,$ $a_{k+1}' = (1/b_k)a_{k+1}(1/b_{k+1})$ and $a_1' = a_1(1/b_1)$, then it is *equivalent* to

$$b_0 + \frac{a_1'}{u} + \frac{a_2'}{u} + \frac{a_3'}{u} + \cdots. \tag{3.3}$$

Let $A_1(z) = a_1z$, $B_1 = b_1$, and, for $n = 2, 3, 4, \ldots,$

$$A_n(z) = a_1\left(b_2 + \frac{a_3}{b_3} + \cdots + \frac{a_n}{b_n}\right)\cdots\left(b_{n-1} + \frac{a_n}{b_n}\right)b_n z,$$

(3.4)

$$B_n = b_n\left(b_{n-1} + \frac{a_n}{b_n}\right)\cdots\left(b_1 + \frac{a_2}{b_2} + \cdots + \frac{a_n}{b_n}\right).$$

Notice that if $S = E^2$, $u = (1, 0)$, and $b_0 = 0$, then $A_n(u)$ and B_n are precisely the nth numerator and denominator, respectively, of (1.3). The only reason for defining A_n as a function is to insure the correct association; it is always evaluated at either u or at $1/B_n$. With the convention that should the formal expression $0 \cdot \infty$ occur, we will interpret it to be u, we have

$$T_1 T_2 \cdots T_n(0) = A_n(1/B_n). \tag{3.5}$$

Thus, the nth approximant of (1.3) is $b_0 + A_n(1/B_n)$.

4. **Some applications**. The continued fraction (1.3) is said to *converge* if and only if not infinitely many terms of the sequence $B_1, B_2, B_3, \ldots$ have value 0 and $A_1(1/B_1), A_2(1/B_2), A_3(1/B_3), \ldots$ converges to a finite limit. (Notice that it is possible for some factor in the expression for B_n to have value 0 and B_n be distinct from 0. In fact, $0(x + y/0)$ has value either 0 or u accordingly as y is or is not 0.)

THEOREM 2. *Suppose that for some positive integer m, $a_m = 0$ while if $m > 1$, $a_n \neq 0$, $n = 1, 2, \ldots, m - 1$. Then* (1.3) *converges if and only if not infinitely many of $B_1, B_2, B_3, \ldots$ are* 0. *When convergent, it has value $A_{m-1}(1/B_{m-1})$.*

Of course it is necessary that not infinitely many of $B_1, B_2, B_3, \ldots$ have value 0. Hence, suppose that N is a positive integer such that, for $m > N$, $B_n \neq 0$. Notice that $B_{m-1} \neq 0$ for if it were 0, we would have that $B_m = B_{m+1} = B_{m+2} = \cdots = 0$. This can be seen in the following manner: Since $a_m = 0$, then B_m is 0 or $b_m B_{m-1}$ accordingly as b_m is or is not 0. Thus, if B_{m-1} is 0, so is B_m. Likewise, B_{m+1} is 0 or $(b_m + a_{m+1}/b_{m+1})B_{m-1}$ accordingly as $b_m + a_{m+1}/b_{m+1}$ is or is not 0. Therefore, when $B_{m-1} = 0$, $B_{m+1} = 0$. This process may be continued.

Suppose that n is a positive integer larger than both N and $m - 1$. Then $B_n \neq 0$ and hence

$$b_m + \frac{a_{m+1}}{b_{m+1}} + \cdots + \frac{a_n}{b_n} \neq 0.$$

Therefore

$$B_n = b_n\left(b_{n-1} + \frac{a_n}{b_n}\right)\cdots\left(b_m + \frac{a_{m+1}}{b_{m+1}} + \cdots + \frac{a_n}{b_n}\right)B_{m-1}$$

and

$$A_n(z) = A_{m-1}\left(b_m + \frac{a_{m+1}}{b_{m+1}} + \cdots + \frac{a_n}{b_n}\right)\cdots\left(b_{n-1} + \frac{a_n}{b_n}\right)b_n z.$$

Thus,

$$A_n(1/B_n) = a_{m-1}(1/B_{m-1})$$

and hence, (1.3) converges to $A_{m-1}(1/B_{m-1})$.

The following identity is closely analogous to one for ordinary continued fractions (see [3, pp. 15–16]).

THEOREM 3. *If no a_k is 0, then, for $n = 1, 2, 3, \ldots,$*

$$\|A_n(1/B_n) - A_{n+1}(1/B_{n+1})\| = \|a_1\| \, \|a_2\| \cdots \|a_{n+1}\| / \|B_n\| \, \|B_{n+1}\|. \tag{4.1}$$

Since no a_k is 0, we may write (1.3) as (3.2). Then according to (2.5) of [1],

$$\|A_n(1/B_n) - A_{n+1}(1/B_{n+1})\| = 1/(D_n D_{n+1})$$

where, for $k = 1, 2, 3, \ldots,$

$$D_k = \|b'_k\| \left\| b'_{k-1} + \frac{1}{b'_k} \right\| \cdots \left\| b'_1 + \frac{1}{b'_2} + \cdots + \frac{1}{b'_k} \right\|.$$

Referring to part (e) of the Lemma, we see that $D_1 = \|1/a_1\| \, \|B_1\|$. Also,

$$D_2 = \|a_1(1/a_2)b_2\| \, \|(1/a_1)b_1 + 1/(a_1(1/a_2)b_2)\|$$

which, by using parts (a), (d), and (e) of the Lemma, can be reduced to $\|1/a_2\| \, \|B_2\|$. By means of induction, we see that for $k = 1, 2, 3, \ldots,$

$$D_{2k-1} = \|(1/a_1)\| \, \|(1/a_3)\| \cdots \|(1/a_{2k-1})\| \, \|B_{2k-1}\|$$

and

$$D_{2k} = \|(1/a_2)\| \, \|(1/a_4)\| \cdots \|(1/a_{2k})\| \, \|B_{2k}\|.$$

From this, (4.1) follows.

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