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OPERATOR RADII OF COMMUTING PRODUCTS

K. OKUBO AND T. ANDO

ABSTRACT. Operator radii $w_\rho(T)$ for a bounded linear operator T on a Hilbert space were introduced in connection with unitary ρ-dilations. We shall be concerned with universal estimates for the ratios

$$w_\rho(ST)/(w_\sigma(S)\cdot w_\rho(T))$$

for commuting operators S, T and $\sigma, \rho > 0$.

1. All operators in this paper are bounded linear operators on a complex Hilbert space $\mathfrak{H}$. We say an operator T belongs to the class $\mathcal{C}_\rho$ $(0 < \rho < \infty)$ if there exists a unitary operator U on some Hilbert space $\mathfrak{K}$ such that $\mathfrak{K}$ contains $\mathfrak{H}$ as a subspace and such that

$$T^n h = \rho P U^n h \quad \text{for } h \in \mathfrak{H} \text{ and } n = 1, 2, \ldots,$$

where P is the orthogonal projection of $\mathfrak{K}$ onto $\mathfrak{H}$. The classes $\mathcal{C}_\rho$ were defined by Sz.-Nagy and Foiaş [7] while Holbrook [4] introduced the *operator radii* $w_\rho(T)$ of an operator T, relative to $\mathcal{C}_\rho$, by the formula:

$$w_\rho(T) = \inf\{\gamma;\, \gamma > 0,\, \gamma^{-1}T \in \mathcal{C}_\rho\}.$$

The family of operator radii includes the familiar quantities in operator theory: $w_1(T) = \|T\|$ (*norm* of T), $w_2(T) = w(T) := \sup\{|(Th, h)|;\ \|h\| = 1\}$ (*numerical radius* of T), and $\lim_{\rho\to\infty} w_\rho(T) = r(T)$ (*spectral radius* of T).

For each $\rho > 0$ the operator radius $w_\rho(\cdot)$ is a pseudonorm on $\mathfrak{B}(\mathfrak{H})$, the space of all operators, in the sense that

$$w_\rho(\alpha T) = |\alpha| w_\rho(T), \qquad w_\rho(T + S) \leqslant \gamma_\rho\{w_\rho(T) + w_\rho(S)\}$$

where γ_ρ is a positive constant depending only on ρ. The constant γ_ρ can be equal to 1 or ρ according as $0 < \rho \leqslant 2$ or $2 < \rho < \infty$. Each operator S induces a linear map $\mathfrak{s}$ on $\mathfrak{B}(\mathfrak{H})$ by the relation: $\mathfrak{s}(T) = ST$. When the space $\mathfrak{B}(\mathfrak{H})$ is provided with the operator radius $w_\rho(\cdot)$, then for each operator S and each $\sigma > 0$ the Lipschitz constant of the map $\mathfrak{s}$ with respect to this pseudonorm $w_\rho(\cdot)$ is majorated by $\sigma(2-\rho)\cdot w_\sigma(S)$ or $\sigma\rho\cdot w_\sigma(S)$ according as $0 < \rho \leqslant 1$ or $1 < \rho < \infty$ (see the next section):

$$w_\rho(ST) \leqslant \begin{cases} \sigma(2-\rho)\cdot w_\sigma(S)\cdot w_\rho(T) & \text{for } 0 < \rho \leqslant 1, \\ \sigma\rho\cdot w_\sigma(S)\cdot w_\rho(T) & \text{for } 1 < \rho < \infty. \end{cases}$$

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We can expect, however, to have better estimates for these if the map $\circ$ is confined to the commutant of S, the subspace of all operators commuting with S. Indeed, it has been conjectured (cf. [4], [5]) that if S commutes with T then

$$(*) \qquad w_\rho(ST) \leqslant \sigma \cdot w_\sigma(S) \cdot w_\rho(T)$$

and, in particular,

$$(**) \qquad w_\rho(ST) \leqslant \|S\| w_\rho(T).$$

Remark that the inequality $(**)$ implies, in its turn, the inequality $(*)$ because of the general relation: $\|S\| \leqslant \sigma \cdot w_\sigma(S)$ for $0 < \sigma < \infty$.

Holbrook [4] and, independently, Sz.-Nagy [6] showed that if S *double commutes* with T, i.e. $ST = TS$ and $ST^* = T^*S$ then $(**)$ holds.

Our purpose in this paper is to take a step towards the above inequalities. We shall prove that if S commutes with T then $(*)$ is valid for $\sigma \geqslant 2$, and for $0 < \sigma < 2$

$$w_\rho(ST) \leqslant L_\sigma \cdot w_\sigma(S) \cdot w_\rho(T)$$

where L_σ is an increasing function of σ such that $L_1 = \sqrt{2}$, $\lim_{\sigma\to 0} L_\sigma = 0$ and $\lim_{\sigma\to 2} L_\sigma = 2$.

As to the second inequality $(**)$, we shall show that if S commutes with T then

$$w_\rho(ST) \leqslant K_\rho \cdot \|S\| \cdot w_\rho(T)$$

where K_ρ is an explicitly given constant such that $0 < K_\rho \leqslant \rho$, $K_1 = 1$ and K_ρ increases to $\sqrt{2}$ as $\rho \to \infty$.

2. First we recall some properties of operator radii used in this paper :

(i) (Sz.-Nagy and Foiaş [7]). $w_\rho(T) \leqslant 1$ if and only if $r(T) \leqslant 1$ and

$$\mathcal{R}e\left[\rho + 2zT(1 - zT)^{-1}\right] \geqslant 0 \quad \text{for } |z| < 1.$$

(ii) (Holbrook [4]). $w_\rho(\xi T) = |\xi| \cdot w_\rho(T)$ for all complex ξ, $w_\rho(T) = w_\rho(T^*)$, and $w_\rho(T)$ is a continuous nonincreasing function of ρ.

(iii) (Ando and Nishio [1]). $\rho \cdot w_\rho(T) = (2 - \rho) \cdot w_{2-\rho}(T)$ for $0 < \rho < 2$, and $\rho \cdot w_\rho(T)$ is nondecreasing for $1 \leqslant \rho < \infty$.

Since $\|S\| \leqslant \sigma \cdot w_\sigma(S)$, $\|T\| \leqslant \rho \cdot w_\rho(T)$ by (ii) and (iii) and $\|ST\| \geqslant w_\rho(ST)$ for $1 \leqslant \rho < \infty$ by (ii), we have the following trivial estimate:

$$w_\rho(ST) \leqslant \|ST\| \leqslant \|S\| \cdot \|T\| \leqslant \sigma\rho \cdot w_\rho(S) \cdot w_\rho(T) \quad \text{for } 1 < \rho < \infty.$$

If $0 < \rho \leqslant 1$ we can apply (iii) to get

$$w_\rho(ST) \leqslant \sigma(2 - \rho) \cdot w_\sigma(S) \cdot w_\rho(T) \quad \text{for } 0 < \rho \leqslant 1.$$

LEMMA 1. *If* $1 \leqslant \rho < \infty$, *then* $w_\rho(T) \leqslant 1$ *implies*

$$(1) \qquad \sum_{n=1}^{\infty} \alpha^{2n} \|T^n h\|^2 \leqslant \frac{\rho\alpha^2 \|h\|^2}{(1 - \alpha)(2 - \rho + \alpha\rho)} \quad \textit{for } h \in \mathfrak{H} \textit{ and } 1 - 2/\rho < \alpha < 1.$$

PROOF. $1 - zT$ has bounded inverse for $|z| < 1$ and $\rho + 2zT(1 - zT)^{-1}$ has positive real part by (i). It is well known that an operator S has positive real part if and only if for each (and all) $\lambda > 0$ the operator $S + \lambda$ has bounded inverse and $(S - \lambda)(S + \lambda)^{-1}$ has norm not greater than one. Applying this to $\rho + 2zT(1 - zT)^{-1}$, we obtain

$$\left\| \left\{ \rho - \lambda + 2zT(1 - zT)^{-1} \right\} \left\{ \rho + \lambda + 2zT(1 - zT)^{-1} \right\}^{-1} \right\| \leqslant 1, \tag{2}$$

which implies

$$\begin{aligned} &\left\| \{ (\rho - \lambda) - (\rho - \lambda - 2)zT \} \{ (\rho + \lambda) - (\rho + \lambda - 2)zT \}^{-1} h \right\| \\ &\qquad \leqslant \|h\| \quad \text{for } h \in \mathfrak{H}. \end{aligned} \tag{3}$$

Since

$$\{ (\rho + \lambda) - (\rho + \lambda - 2)zT \}^{-1} = (\rho + \lambda)^{-1} \sum_{n=0}^{\infty} \left(\frac{\rho + \lambda - 2}{\rho + \lambda} zT \right)^n,$$

we have, for $\lambda \neq \rho$,

$$\begin{aligned} &\{ (\rho - \lambda) - (\rho - \lambda - 2)zT \} \{ (\rho + \lambda) - (\rho + \lambda - 2)zT \}^{-1} \\ &\qquad = \frac{\rho - \lambda}{\rho + \lambda} \left\{ 1 + \frac{4\lambda}{(\rho - \lambda)(\rho + \lambda - 2)} \sum_{n=1}^{\infty} \left(\frac{\rho + \lambda - 2}{\rho + \lambda} zT \right)^n \right\}. \end{aligned}$$

Therefore, setting $z = re^{i\theta}$ and integrating the squares of both sides of (3) over $(0, 2\pi]$, we have

$$\begin{aligned} &\frac{1}{2\pi} \int_0^{2\pi} \left\| h + \frac{4\lambda}{(\rho - \lambda)(\rho + \lambda - 2)} \sum_{n=1}^{\infty} \left(\frac{\rho + \lambda - 2}{\rho + \lambda} re^{i\theta} T \right)^n h \right\|^2 d\theta \\ &\qquad \leqslant \left(\frac{\rho + \lambda}{\rho - \lambda} \right)^2 \|h\|^2, \end{aligned}$$

and further, using the orthogonality relation of $e^{in\theta}$ and $e^{im\theta}$ $(n \neq m)$ and letting $r \to 1$,

$$\sum_{n=1}^{\infty} \left(\frac{\rho + \lambda - 2}{\rho + \lambda} \right)^{2n} \|T^n h\|^2 \leqslant \left\{ \left(\frac{\rho + \lambda}{\rho - \lambda} \right)^2 - 1 \right\} \left\{ \frac{(\rho - \lambda)(\rho + \lambda - 2)}{4\lambda} \right\}^2 \|h\|^2,$$

hence

$$\sum_{n=1}^{\infty} \left(\frac{\rho + \lambda - 2}{\rho + \lambda} \right)^{2n} \|T^n h\|^2 \leqslant \frac{\rho(\rho + \lambda - 2)^2}{4\lambda} \|h\|^2. \tag{4}$$

Since λ can vary over $(0, \infty) \setminus \{\rho\}$, $\alpha \equiv (\rho + \lambda - 2)(\rho + \lambda)^{-1}$ can move over $(1 - 2/\rho, 1) \setminus \{1 - 1/\rho\}$. We can conclude, replacing $(\rho + \lambda - 2)(\rho + \lambda)^{-1}$ by α in (4), that (1) is valid except for $\alpha = 1 - 1/\rho$. This exception can, however, be removed by limit procedure.

Remark that the inequality (1) for the special case $\alpha = 1 - 1/\rho$ was obtained by Berger and Stampfli [2] by a different method.

LEMMA 2. *If* $1 < \rho < \infty$ *and*

(5) $$\sum_{n=1}^{\infty} \rho^{-n}(\rho-1)^n|(T^n h, g)| \leqslant (\rho-1)\|h\|\cdot\|g\| \quad \textit{for } h, g \in \mathfrak{H}$$

then $w_\rho(T) \leqslant 1$.

PROOF. First of all, (5) implies $\|T^n\|^{1/n} \leqslant \rho(\rho-1)^{1/n-1}$, hence $r(T) \leqslant \rho(\rho-1)^{-1}$, for the spectral radius $r(T)$ is equal to $\lim_{n\to\infty}\|T^n\|^{1/n}$. Then for each $|z| < 1$ the series $\sum_{n=1}^{\infty}\{\rho^{-1}(\rho-1)zT\}^n$ is uniformly convergent to $\rho^{-1}(\rho-1)zT\{1-\rho^{-1}(\rho-1)zT\}^{-1}$, and by (5)

$$\left|\left(zT\{\rho-(\rho-1)zT\}^{-1}h, g\right)\right| \leqslant \|h\|\cdot\|g\| \quad \text{for } h, g \in \mathfrak{H},$$

or equivalently

(6) $$\left\|zT\{\rho-(\rho-1)zT\}^{-1}h\right\| \leqslant \|h\| \quad \text{for } h \in \mathfrak{H} \text{ and } |z| < 1,$$

which is just (3) with $\lambda = \rho$. If $r(T) \leqslant 1$, then (6) can be converted to (2) with $\lambda = \rho$, hence $w_\rho(T) \leqslant 1$ by (i) as remarked in the proof of Lemma 1. Now let us show that (6) implies $r(T) \leqslant 1$. In fact, if ξ is an approximate eigenvalue such that $|\xi| = r(T) > 1$, take $|z| < 1$ such that $z\xi = 1 + \varepsilon$ for some $\varepsilon > 0$ small enough so that $\rho - (\rho-1)(1+\varepsilon) > 0$. Then (6) implies

$$1 + \varepsilon = |z\xi| \leqslant |\rho - (\rho-1)z\xi| = \rho - (\rho-1)(1+\varepsilon)$$

or $0 \leqslant -\rho\varepsilon$, a contradiction.

THEOREM 1. *If S commutes with T, i.e. $ST = TS$ then*

$$w_\rho(ST) \leqslant L_\sigma \cdot w_\sigma(S)\cdot w_\rho(T) \quad \textit{for } 0 < \sigma,\rho < \infty$$

where

$$L_\sigma = \begin{cases} \dfrac{\sigma - 1 + (1+2\sigma-\sigma^2)^{1/2}}{2-\sigma} & \textit{for } 0 < \sigma \leqslant 1, \\[2ex] \dfrac{1 - \sigma + (1+2\sigma-\sigma^2)^{1/2}}{2-\sigma} & \textit{for } 1 < \sigma < 2, \\[2ex] \sigma & \textit{for } 2 \leqslant \sigma < \infty. \end{cases}$$

PROOF. Since $\sigma\cdot w_\sigma(S) = (2-\sigma)\cdot w_{2-\sigma}(S)$ for $0 < \sigma < 1$ by (iii) and $w_\sigma(\lambda S) = \lambda\cdot w_\sigma(S)$ for $\lambda > 0$ by (ii) and similarly for ρ and T, we may assume that $1 \leqslant \sigma,\rho$ and $w_\sigma(S) = w_\rho(T) = 1$. Given $\beta > 1$, by Lemma 2, a sufficient condition for $w_\rho(ST) \leqslant \beta$ is that

$$\sum_{n=1}^{\infty} \beta^{-n}\rho^{-n}(\rho-1)^n\left|((ST)^n h, g)\right| \leqslant (\rho-1)\|h\|\cdot\|g\| \quad \text{for } h, g \in \mathfrak{H}.$$

Since $ST = TS$ implies

$$\left|((ST)^n h, g)\right| = |(T^n h, S^{*n}g)| \leqslant \|T^n h\|\cdot\|S^{*n}g\|,$$

a sufficient condition for $w_\rho(ST) \leqslant \beta$ is

(7) $$\sum_{n=1}^{\infty} \beta^{-n}\rho^{-n}(\rho-1)^n\|T^n h\|\cdot\|S^{*n}g\| \leqslant (\rho-1)\|h\|\cdot\|g\| \quad \text{for } h, g \in \mathfrak{H}.$$

To get an estimate for the left side of (7), take γ such that

(8) $1 - 2/\rho < 1 - \gamma/\rho < 1$ and $1 - 2/\sigma < (\rho - 1)/\beta(\rho - \gamma) < 1$.

Then we have by Lemma 1

$$\sum_{n=1}^{\infty}\left(1 - \frac{\gamma}{\rho}\right)^{2n}\|T^n h\|^2 \leqslant \frac{(\rho - \gamma)^2}{\gamma(2 - \gamma)}\|h\|^2 \tag{9}$$

and by (ii): $w_\sigma(S^*) = w_\sigma(S) = 1$,

$$\begin{aligned}&\sum_{n=1}^{\infty}\left(\frac{\rho - 1}{\beta(\rho - \gamma)}\right)^{2n}\|S^{*n}g\|^2\\ &\leqslant \frac{\sigma(\rho - 1)^2\|g\|^2}{\{\beta(\rho - \gamma) - (\rho - 1)\}\{(2 - \sigma)\beta(\rho - \gamma) + \sigma(\rho - 1)\}}.\end{aligned} \tag{10}$$

Applying the Schwartz inequality, we obtain from (9) and (10)

$$\begin{aligned}&\left\{\sum_{n=1}^{\infty}\beta^{-n}\rho^{-n}(\rho - 1)^n\|T^n h\|\cdot\|S^{*n}g\|\right\}^2\\ &\leqslant \sum_{n=1}^{\infty}\left(1 - \frac{\gamma}{\rho}\right)^{2n}\|T^n h\|^2\cdot\sum_{n=1}^{\infty}\left(\frac{\rho - 1}{\beta(\rho - \gamma)}\right)^{2n}\|S^{*n}g\|^2\\ &\leqslant \frac{\sigma(\rho - \gamma)^2(\rho - 1)^2\|h\|^2\cdot\|g\|^2}{\gamma(2 - \gamma)\{\beta(\rho - \gamma) - (\rho - 1)\}\{(2 - \sigma)\beta(\rho - \gamma) + \sigma(\rho - 1)\}}.\end{aligned}$$

Now a sufficient condition for $w_\rho(ST) \leqslant \beta$ is that the inequality

$$\begin{aligned}&(2 - \sigma)(\rho - \gamma)^2\beta^2 - 2(\rho - 1)(\rho - \gamma)(1 - \sigma)\beta\\ &\quad - \sigma(\rho - 1)^2 - \sigma\cdot\gamma^{-1}(2 - \gamma)^{-1}(\rho - \gamma)^2 \geqslant 0\end{aligned} \tag{11}$$

has a solution γ satisfying (8), for then (7) holds.

Now if $1 \leqslant \sigma \leqslant 2$ and $\rho > 1$, then (8) is satisfied with $\gamma = 1$ and (11) takes the following form

$$(2 - \sigma)\beta^2 - 2(1 - \sigma)\beta - 2\sigma \geqslant 0$$

or equivalently

$$\beta \geqslant \begin{cases}\dfrac{1 - \sigma + (1 + 2\sigma - \sigma^2)^{1/2}}{2 - \sigma} & \text{for } 1 \leqslant \sigma < 2,\\ 2 & \text{for } \sigma = 2.\end{cases}$$

This proves the assertion for $0 < \sigma \leqslant 2$. Finally the assertion for $\sigma > 2$ results from the above by (iii), i.e. $\sigma \cdot w_\sigma(S) \geqslant 2 \cdot w_2(S)$ for $\sigma > 2$.

It is quite easy to show that L_σ is a nondecreasing function of σ on $[1, \infty)$ while L_σ/σ decreases to 1 as σ increases to 2.

Holbrook [4, Theorem 6.3] proved that if S commutes with T then

$$w_\rho(ST) \leqslant 2w_\rho(S)\cdot w_\rho(T) \quad \text{for } 0 < \rho \leqslant 2.$$

Theorem 1 gives an improvement because $L_\sigma < 2$ for $0 < \sigma < 2$. On the other hand, under the assumption of double commutativity, i.e. $ST = TS$ and $ST^* = T^*S$, Holbrook [4, Theorem 6.2] proved the inequality:

$$w_\rho(ST) \leqslant \rho \cdot w_\rho(S) \cdot w_\rho(T) \quad \text{for } 0 < \rho < \infty.$$

Our Theorem 1 shows that double commutativity can be replaced by mere commutativity for $2 \leqslant \rho < \infty$.

3. As remarked in the preceding section,

$$w_\rho(ST) \leqslant \rho \cdot \|S\| \cdot w_\rho(T) \quad \text{for } 1 \leqslant \rho < \infty.$$

Holbrook [4], [5] pointed out that for each $\rho > 1$ there is a constant $K_\rho < \rho$ such that if S commutes with T

$$w_\rho(ST) \leqslant K_\rho \cdot \|S\| \cdot w_\rho(T).$$

Further he showed that if we confine ourselves to commuting operators on a Hilbert space of fixed finite dimension then K_ρ can be chosen so that $K_\rho/\rho \to 0$ as $\rho \to \infty$. But he could not get any explicit form of K_ρ. The following theorem gives an estimate for K_ρ and, at the same time, removes the restriction on dimension.

THEOREM 2. *If S commutes with T, then*

$$w_\rho(ST) \leqslant K_\rho \cdot \|S\| \cdot w_\rho(T) \quad \textit{for } 0 < \rho < \infty,$$

where

$$K_\rho = \begin{cases} \inf\limits_{0<\gamma<1} \left\{ \dfrac{1}{\gamma(2-\gamma)} + \dfrac{(\rho-1)^2}{(2-\rho-\gamma)^2} \right\}^{1/2} & \textit{for } 0 < \rho < 1, \\[2ex] \inf\limits_{0<\gamma<1} \left\{ \dfrac{1}{\gamma(2-\gamma)} + \dfrac{(\rho-1)^2}{(\rho-\gamma)^2} \right\}^{1/2} & \textit{for } 1 \leqslant \rho < \infty. \end{cases}$$

PROOF. As in the proof of Theorem 1, we may assume that $\|S\| = w_\rho(T) = 1$ and $\rho > 1$. Now (8) with $\sigma = 1$ is reduced to

$$(12) \qquad 0 < \gamma < \min\{2, \rho - \beta^{-1}(\rho - 1)\},$$

while (11) takes the form:

$$(13) \qquad (\rho-\gamma)^2\beta^2 - (\rho-1)^2 - \gamma^{-1}(2-\gamma)^{-1}(\rho-\gamma)^2 \geqslant 0.$$

Since every $0 < \gamma < 1$ satisfies (12), we get the assertion from (13), just as in the proof of Theorem 1.

Since, for each fixed γ with $0 < \gamma < 1$, $(\rho - 1)(\rho - \gamma)^{-1}$ is an increasing function of $1 \leqslant \rho < \infty$, the function K_ρ is increasing and satisfies

$$K_1 = 1, \quad K_2 = \sqrt{27}/4 \quad \text{and} \quad \lim_{\rho\to\infty} K_\rho = \sqrt{2}$$

and

$$\lim_{\rho\to\infty} K_\rho/\rho = 0.$$

Obviously $K_\rho \leqslant \sqrt{2} < \rho$ for $2 \leqslant \rho < \infty$. Let $1 < \rho < 2$. Setting $\gamma = \rho - (\rho-1)^{1/2}$ in $1/\gamma(2-\gamma) + (\rho-1)^2/(\rho-\gamma)^2$, we see

$$K_\rho^2 \leqslant \rho - 1 + \left\{\rho - (\rho-1)^{1/2}\right\}^{-1}\left\{2 - \rho + (\rho-1)^{1/2}\right\}^{-1} < \rho^2.$$

4. When specialized to $\rho = 2$, the most important and interesting case, Theorem 2 tells us that if S commutes with T then

$$w(ST) \leqslant \left(\sqrt{27}/4\right) \cdot \|S\| \cdot w(T).$$

Of course, $\sqrt{27}/4 \simeq 1.2990$ is not at all the best possible estimate for K_2. In fact, M. J. Crabb informed us, in a private communication, of an effective method of getting better estimates for K_2. Let us sketch his idea.

Suppose that $ST = TS$, $\|S\| = w(T) = 1$ and h is a unit vector. Let

$$\alpha_n = \|T^n h\|^2 + \|T^{*n} h\|^2 \qquad (n = 1, 2, \ldots)$$

and

$$\beta_n = \alpha_{2^{n-1}} \qquad (n = 1, 2, \ldots).$$

By the Schwartz inequality

$$\begin{aligned} 4|\mathcal{R}e(STh, h)|^2 &= |(STh + T^*S^*h, h)|^2 \leqslant \|STh + T^*S^*h\|^2 \\ &= \|STh\|^2 + \|T^*S^*h\|^2 + 2\,\mathcal{R}e\big((ST)^2 h, h\big). \end{aligned}$$

Since $ST = TS$ and $\|S\| \leqslant 1$, we have

$$\begin{aligned} 4|\mathcal{R}e(STh, h)|^2 &\leqslant \|Th\|^2 + \|T^*h\|^2 + 2|\mathcal{R}e(S^2T^2h, h)| \\ &\leqslant \beta_1 + 2|\mathcal{R}e(S^2T^2h, h)|. \end{aligned}$$

Applying the same method to S^2T^2 and so on, we have

$$\begin{aligned} &2|\mathcal{R}e(STh, h)| \\ &\leqslant \Big\{ \beta_1 + \Big\{ \beta_2 + \Big\{ \beta_3 + \cdots \\ &\qquad + \big\{ \beta_n + 2|\mathcal{R}e(S^{2^n}T^{2^n}h, h)| \big\}^{1/2} \Big\}^{1/2} \Big\}^{1/2} \cdots \Big\}^{1/2} \\ &\leqslant \Big\{ \beta_1 + \Big\{ \beta_2 + \Big\{ \beta_3 + \cdots + \{2\beta_n\}^{1/2} \Big\}^{1/2} \Big\}^{1/2} \cdots \Big\}^{1/2} \end{aligned}$$

because

$$\begin{aligned} 2|\mathcal{R}e(S^{2^n}T^{2^n}h, h)| &\leqslant 2\|S^{2^{n-1}}T^{2^{n-1}}h\| \cdot \|S^{*2^{n-1}}T^{*2^{n-1}}h\| \\ &\leqslant \|T^{2^{n-1}}h\|^2 + \|T^{*2^{n-1}}h\|^2 = \beta_n. \end{aligned}$$

Now $w(T) \leqslant 1$ implies

$$\mathcal{R}e(e^{i\theta}Tg, g) \leqslant (g, g) \quad \text{for } g \in \mathfrak{H}.$$

Integrating this inequality over $0 < \theta \leqslant 2\pi$ with

$$g = \xi_2 e^{-2i\theta} T^{*2} h + \xi_1 e^{-i\theta} T^* h + \xi_0 h + \xi_1 e^{i\theta} Th + \xi_2 e^{2i\theta} T^2 h$$

and using the orthogonality of $\{e^{ik\theta}\}$ we obtain

$$\mathcal{R}e\big(\xi_0\bar{\xi}_1\alpha_1 + \xi_1\bar{\xi}_2\alpha_2\big) \leqslant |\xi_0|^2 + |\xi_1|^2\alpha_1 + |\xi_2|^2\alpha_2 \tag{14}$$

for arbitrary complex numbers ξ_k. An elementary argument reveals that (14) simply asserts $\alpha_2 \leqslant \alpha_1(4 - \alpha_1)$. Hence, noting that $\beta_1 = \alpha_1$ and $\beta_2 = \alpha_2$, we are led to the inequalities:

$$\begin{aligned} |\mathcal{R}e(STh, h)| &\leqslant \tfrac{1}{2}\left\{\alpha_1 + \{2\alpha_2\}^{1/2}\right\}^{1/2} \\ &\leqslant \tfrac{1}{2}\left\{\alpha_1 + \sqrt{2}\,\{\alpha_1(4 - \alpha_1)\}^{1/2}\right\}^{1/2} \\ &\leqslant \max_{0 \leqslant x \leqslant 4} \tfrac{1}{2}\left\{x + \sqrt{2}\,\{x(4 - x)\}^{1/2}\right\}^{1/2} \\ &= \tfrac{1}{2}\{2 + 2\sqrt{3}\}^{1/2} \simeq 1.169. \end{aligned}$$

Since $e^{i\theta}S$, for any θ, may replace S in the above, we obtain $w(ST) \leqslant 1.169$. This method may be extended to obtain, in principle, a sequence of estimates for $w(ST)$. It is easy to generalize the argument leading to (14) to obtain

$$\mathcal{R}e\left\{-\sum_{k=1}^{n} \xi_{k-1}\bar{\xi}_k\alpha_k\right\} \leqslant |\xi_0|^2 + \sum_{k=1}^{n} |\xi_k|^2\alpha_k \tag{15}$$

for an arbitrary complex sequence $\{\xi_k\}$. Then as in Crabb [3] we may conclude that

$$\alpha_k = 2\left\{\gamma_{k-1} \pm \left(\gamma_{k-1}^2 - \gamma_{k-1}\gamma_k\right)^{1/2}\right\}$$

for some sequence

$$1 = \gamma_0 \geqslant \gamma_1 \geqslant \cdots \geqslant 0.$$

Letting $\delta_0 = 1$ and $\delta_k = \gamma_{2^{k-1}}$, we have

$$\beta_k = \alpha_{2^{k-1}} \leqslant 2\left\{\delta_{k-1} + \left(\delta_{k-1}^2 - \delta_{k-1}\delta_k\right)^{1/2}\right\},$$

so that, for each n, $w(ST)$ is bounded above by the maximum of

$$\tfrac{1}{2}\left\{2\delta_0 + 2(\delta_0^2 - \delta_0\delta_1)^{1/2} + \left\{\cdots + \left\{4\delta_{n-1} + 4(\delta_{n-1}^2 - \delta_{n-1}\delta_n)^{1/2}\right\}^{1/2} \cdots\right\}^{1/2}\right\}$$

subject to the restriction:

$$1 = \delta_0 \geqslant \delta_1 \geqslant \cdots \geqslant \delta_n \geqslant 0.$$

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RESEARCH INSTITUTE OF APPLIED ELECTRICITY, HOKKAIDO UNIVERSITY, SAPPORO, JAPAN