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ON THE STRUCTURE OF LINDENBAUM ALGEBRAS: AN APPROACH USING ALGEBRAIC LOGIC

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ABSTRACT. The following problem of algebraic logic is investigated: to determine those Boolean algebras which admit the structure of a nondiscrete cylindric algebra. A partial solution is found, and is then used to give an algebraic characterization of the Lindenbaum algebras of formulas of several broad classes of countable theories.

1. **Introduction.** A major open problem of algebraic logic is the following: Which Boolean algebras admit the structure of a nondiscrete cylindric or polyadic algebra? Using results of Henkin, Monk and Tarski [1], one easily proves:

A denumerable Boolean algebra admits the structure of a nondiscrete, dimension-complemented cylindric algebra if and only if it is not atomic.

We establish this, as well as a few related results, and use them to investigate the structure of Lindenbaum algebras of countable theories.

In the sequel, let T denote any countable theory. By the Lindenbaum algebra $\mathfrak{F}_T$ of T we will always mean the Lindenbaum algebra of *formulas*[1] of T. From results in [1] we easily establish that if T has no one-element models then $\mathfrak{F}_T$ is atomless (this characterizes $\mathfrak{F}_T$, for there is, up to isomorphism, only one atomless, denumerable Boolean algebra). Let us say that *T admits elimination of all but n predicates* if T is definitionally equivalent to a theory T' whose language may have finitely or denumerably many operation symbols, but has no more than n predicate symbols other than $=$; for $n = 0$, we say that *T admits elimination of predicates*. It is shown that an arbitrary theory T admits elimination of predicates if and only if T has either no one-element models, or all of its one-element models are elementarily equivalent. We prove that if a theory T admits elimination of predicates then $\mathfrak{F}_T$ has $\leqslant 1$ atom; more generally, if T admits elimination of all but n predicates then $\mathfrak{F}_T$ has $\leqslant 2^n$ atoms. Then we provide a method to determine the exact structure of $\mathfrak{F}_T$ whenever T admits elimination of all but finitely many predicates.

Our notation and terminology is that of Henkin, Monk and Tarski [1], and we presuppose an acquaintance at least with Chapter 1 of this work.

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[1] We will deal here with Lindenbaum algebras of formulas, rather than Lindenbaum algebras of sentences.

2. **Results on cylindric algebras.** Throughout this section, let $\mathfrak{A} = \langle A, +, \cdot, -, 0, 1, c_\kappa, d_{\kappa\lambda} \rangle_{\kappa,\lambda<\alpha}$ be a nondiscrete, dimension-complemented cylindric algebra. By [1, 1.11.3(iii)], $\alpha \geqslant \omega$. The following statements, which are easily deduced from results given in [1], will be needed in the sequel:

(A) if $c_0^\partial d_{01} = 0$, then $\mathfrak{A}$ is atomless;

(B) if $c_0^\partial d_{01} \neq 0$, and there is no zero-dimensional element $x \neq 0$ such that $x < c_0^\partial d_{01}$, then $c_0^\partial d_{01}$ is the only atom of $\mathfrak{A}$;

(C) $\mathfrak{A} \cong \mathfrak{B} \times \mathfrak{C}$ where $\mathfrak{B}$ is atomless and $\mathfrak{C}$ is discrete.

(A) and (B) follow from [1, Theorems 1.10.5(ii), 1.11.8(i) and 1.6.20$^\partial$]. (C) follows from [1, Theorems 2.4.37 and 1.11.8(ii)].

Every Boolean algebra admits the structure of a *discrete* cylindric algebra, so there is no need to consider that case further. Similarly, every Boolean algebra admits the structure of a cylindric algebra of degree 1, for example by taking c to be the quantifier given by $c0 = 0$, $x \neq 0 \Rightarrow cx = 1$. Thus, we should confine our attention to nondiscrete cylindric algebras of degree $\alpha \geqslant 2$.

If $\mathfrak{B}$ is any denumerable, atomless Boolean algebra, then $\mathfrak{B}$ admits the structure of a nondiscrete cylindric algebra of degree ω. Indeed, if T is any countable theory which has no one-element models, then $T \vdash \neg(\forall v_0)(v_0 = v_1)$, hence by (A), the Lindenbaum algebra of formulas of T is an atomless denumerable Boolean algebra. This Lindenbaum algebra is isomorphic to $\mathfrak{B}$ because any two denumerable atomless Boolean algebras are isomorphic.

Now, let $\mathfrak{A}$ be any denumerable Boolean algebra having a direct factor which is an atomless denumerable Boolean algebra, say $\mathfrak{A} \cong \mathfrak{B} \times \mathfrak{C}$ where $\mathfrak{B}$ is atomless and denumerable. We have just seen that $\mathfrak{B}$ admits the structure of a nondiscrete, dimension-complemented cylindric algebra, and $\mathfrak{C}$ certainly admits the structure of a discrete cylindric algebra, hence $\mathfrak{A}$ admits the structure of a nondiscrete, dimension-complemented cylindric algebra. Combining this with (C), we get

(D) A denumerable Boolean algebra admits the structure of a nondiscrete dimension-complemented cylindric algebra if and only if it has a direct factor which is denumerable and atomless.

By the elementary theory of Boolean algebras, to say that $\mathfrak{A}$ *has a direct factor which is denumerable and atomless* is equivalent to saying that $\mathfrak{A}$ is not atomic. Thus, we have proved

THEOREM 1. *A denumerable Boolean algebra admits the structure of a nondiscrete dimension-complemented cylindric algebra if and only if it is not atomic.*

In the discussion which follows we will use an algebraic counterpart of terms in first-order languages. For a full discussion of terms in cylindric algebras the reader is referred to [4]; however, for the present purposes only a few rudimentary notions are needed. An element $x \in A$ will be called "diagonal-like" if it has the following two properties for some $\kappa < \alpha$:

(1) $c_\kappa x = 1$, and

(2) $x \cdot s_\mu^\kappa x \leqslant d_{\kappa\mu}$ for each $\mu \in \alpha - \Delta x$.

With every diagonal-like element $x \in A$ we associate a term a, and (for x satisfying (1) and (2) above), we write $x = d_{\kappa a}$. (In the metalogical interpretation, $d_{\kappa a}$ is the equivalence class of the formula $v_\kappa = a$, and (1) and (2) assert the unique existence of v_κ satisfying $v_\kappa = a$. Thus, if $\mathfrak{A}$ is taken to be an algebra of formulas, the "terms" of $\mathfrak{A}$ are all the terms which are explicitly definable in the theory associated with $\mathfrak{A}$.)

The following properties of diagonal-like elements will be relevant to our discussion:

THEOREM 2. *If x is any diagonal-like element, then $x \geqslant c_0^\partial d_{01}$.*

PROOF. Let x satisfy (1) and (2). By [1, 1.6.20], $c_\kappa^\partial d_{\kappa\lambda} - x \in Zd\mathfrak{A}$. Thus,

$$c_\kappa^\partial d_{\kappa\lambda} \cdot -x = c_\kappa^\partial(c_\kappa^\partial d_{\kappa\lambda} \cdot -x) = c_\kappa^\partial d_{\kappa\lambda} \cdot -c_\kappa x = c_\kappa^\partial d_{\kappa\lambda} \cdot 0 = 0.$$

Thus, $c_\kappa^\partial d_{\kappa\lambda} \leqslant x$.

From this theorem, we deduce a useful generalization of [1, Theorem 2.3.33]:

COROLLARY 3. *Suppose $\mathfrak{A} \in Dc_\alpha$, $\alpha \geqslant 2$, and $\mathfrak{A}$ has a set of generators, X, such that all but n elements of X are diagonal-like. Then*

(i) $|At\mathfrak{A}| \leqslant 2^n$, *and*

(ii) $c_0^\partial d_{01} = \sum At\mathfrak{A}$.

PROOF. By Theorem 2, if x is diagonal-like, then $x \cdot c_0^\partial d_{01} = c_0^\partial d_{01}$. The remainder of the argument is exactly as in [1, Theorems 2.3.31 and 2.3.33].

The converse of Theorem 2, which follows next, states that if $x \geqslant c_0^\partial d_{01}$, then x is generated from the diagonal-like elements of A .

THEOREM 4. *If $x \geqslant c_0^\partial d_{01}$, then there is a diagonal-like element y such that $x = -c_\kappa c_\lambda c_\mu(y \cdot d_{\kappa\mu} - d_{\kappa\lambda})$.*

PROOF. Take distinct $\kappa, \lambda, \mu \in \alpha - \Delta x$. Let

$$y = d_{\kappa\lambda} \cdot d_{\kappa\mu} + d_{\kappa\mu} - d_{\kappa\lambda} - x + d_{\lambda\mu} - d_{\kappa\lambda} \cdot x.$$

One verifies directly (we omit the simple details) that $c_\mu y = 1$, and for any $\nu \in \alpha - \Delta y$, $y \cdot s_\nu^\mu y \leqslant d_{\mu\nu}$. Thus, y is a diagonal-like element. We note that

$$c_\kappa c_\lambda c_\mu(-d_{\kappa\lambda} \cdot y \cdot d_{\kappa\mu}) = c_\kappa c_\lambda[-d_{\kappa\lambda} \cdot c_\mu(y \cdot d_{\kappa\mu})] = c_\kappa c_\lambda(-d_{\kappa\lambda} \cdot s_\kappa^\mu y).$$

Now, $s_\kappa^\mu y = d_{\kappa\lambda} + -d_{\kappa\lambda} \cdot -x$, hence

$$\begin{aligned} c_\kappa c_\lambda(-d_{\kappa\lambda} \cdot s_\kappa^\mu y) &= c_\kappa c_\lambda(-d_{\kappa\lambda} \cdot [d_{\kappa\lambda} + -d_{\kappa\lambda} \cdot -x]) \\ &= c_\kappa c_\lambda(-d_{\kappa\lambda} \cdot -x) = (c_\kappa c_\lambda - d_{\kappa\lambda}) \cdot -x. \end{aligned}$$

But by assumption, $-x \leqslant -c_0^\partial d_{01} = c_\kappa c_\lambda - d_{\kappa\lambda}$, so $c_\kappa c_\lambda(-d_{\kappa\lambda} \cdot s_\kappa^\mu y) = -x$.

COROLLARY 5. *$\mathfrak{A}$ is generated by its diagonal-like elements iff $c_0^\partial d_{01} = 0$ or $c_0^\partial d_{01}$ is an atom.*

PROOF. If X is a set of generators of $\mathfrak{A}$, then (as in the proof of [1, 2.3.31],

$Rl_{c_0^\partial d_{01}}\mathfrak{A}$ is generated by $\{x \cdot c_0^\partial d_{01} : x \in X\}$. Thus, if X contains only diagonal-like elements, then by Theorem 2, $\{x \cdot c_0^\partial d_{01} : x \in X\} = \{c_0^\partial d_{01}\}$, hence $c_0^\partial d_{01} = 0$ or $c_0^\partial d_{01}$ is an atom. Conversely, suppose that $c_0^\partial d_{01} = 0$ or $c_0^\partial d_{01}$ is an atom. In the first case, $x \geqslant c_0^\partial d_{01}$ for every $x \in A$; in the second case, either $x \geqslant c_0^\partial d_{01}$ or $-x \geqslant c_0^\partial d_{01}$ for every $x \in A$. Thus, by Theorem 4, $\mathfrak{A}$ is generated by its diagonal-like elements.

Finally, the following result is of some interest:

THEOREM 6. *$\mathfrak{A}$ has a set of generators, X, which contains only diagonal-like elements and zero-dimensional elements.*

PROOF. $\mathfrak{A} \cong Rl_{c_0^\partial d_{01}}\mathfrak{A} \times Rl_{-c_0^\partial d_{01}}\mathfrak{A}$; as we have seen above, $Rl_{-c_0^\partial d_{01}}\mathfrak{A}$ is generated by its diagonal-like elements, and $Rl_{c_0^\partial d_{01}}\mathfrak{A}$ is generated by zero-dimensional elements.

3. **Applications to Lindenbaum algebras.** The connections between algebraic logic and logic are studied in [2] and [3]. For example, it is proved in [3] that two arbitrary theories are definitionally equivalent iff their associated cylindric algebras are isomorphic. Furthermore, from the discussion in [3], it is clear that if a language has no relation symbols, then its associated cylindric algebra is generated by its diagonal-like elements; and conversely, if the cylindric algebra associated with a theory T is generated by its diagonal-like elements, then T is definitionally equivalent to a theory in a language with no relation symbols. In the sequel, these facts will be used without further explicit mention.

Throughout this section, we will take $\mathfrak{A}$ to be the Lindenbaum algebra of formulas, $\mathfrak{F}_T$, of a first-order theory T.

We will show, first, that Corollary 5 yields a necessary and sufficient condition for the eliminability of predicates in favor of functions. We begin by noting the following:

For $\mathfrak{A} = \mathfrak{F}_T$, $c_0^\partial d_{01} = 0$ iff $T \vdash \neg(\forall v_0)\ (v_0 = v_1)$ iff T has no one-element models. On the other hand, $c_0^\partial d_{01}$ is an atom if and only if $Rl_{c_0^\partial d_{01}}\mathfrak{A}$ is a two-element Boolean algebra. Now, the discrete cylindric algebra $Rl_{c_0^\partial d_{01}}\mathfrak{A}$ is the algebra of formulas of the theory of one-element models of T, and a theory is complete iff its Boolean algebra of sentences has two elements; thus, $c_0^\partial d_{01}$ is an atom iff all the one-element models of T are elementarily equivalent. Combining this with Corollary 5, we get

(E) *An arbitrary theory T admits elimination of predicates iff T either has no one-element models, or all of its one-element models are elementarily equivalent.*

From Theorem 6 we deduce that for any theory T, predicate symbols are eliminable in favor of function symbols and propositional constants:

(F) *Any theory T is definitionally equivalent to a theory T' whose language has no predicate symbols but may have function symbols and propositional constants.*

(Note that the propositional constants serve only to axiomatize the class of one-element models of T.)

We will now use Corollary 3 to describe the structure of the Lindenbaum algebras of formulas of certain theories. If T has no one-element models, then, as we have already noted, in $\mathfrak{F}_T$, $c_0^\partial d_{01} = 0$. Thus, in view of (A), we have

(G) *If T has no one-element models, then $\mathfrak{F}_T$ is an atomless Boolean algebra.*

If T *has* one-element models, we may use Corollary 3 to deduce

(H) *If T admits elimination of all but n predicates, then $\mathfrak{F}_T$ has $\leqslant 2^n$ atoms.*

If T is any theory which admits elimination of all but finitely many predicates we can, in fact, find the exact structure of $\mathfrak{F}_T$. We assume the language L of T is denumerable.

Let $\langle P_i \rangle_{i<n}$ be the sequence of predicate symbols of L, and let δ_i be the rank of P_i for each $i < n$. It follows from [1, Theorem 2.4.37] that $\mathfrak{F}_T \cong \mathfrak{B} \times \mathfrak{C}$, where $\mathfrak{B} = Rl_{c\mathfrak{d} d_{01}} \mathfrak{A}$ and $\mathfrak{C} = Rl_{-c\mathfrak{d} d_{01}} \mathfrak{A}$. We have already seen that $\mathfrak{C}$ belongs to the isomorphism class of denumerable, atomless Boolean algebras, so it remains only to determine the structure of $\mathfrak{B}$. Now, $\mathfrak{B}$ is the algebra of formulas of the theory T_1 whose nonlogical axioms are those of T together with the formula $(\forall v_0)\ (v_0 = v_1)$. It is immediately verified that

$$T_1 \vdash P_i(v_1, \ldots, v_{\delta_i}) \leftrightarrow P_i(t_1, \ldots, t_{\delta_i})$$

for every $i < n$ and all terms t_j, and $T_1 \vdash (\forall v_\kappa)F \leftrightarrow F$ for every formula F. Thus $\mathfrak{B}$, the algebra of formulas of T_1, is the same as the algebra of formulas of the theory in the propositional calculus whose propositional variables are $P_1, \ldots, P_{n-1}$, and whose axioms are obtained from those of T_1 by deleting all variables, terms and quantifiers (together with the associated commas and brackets).

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