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ISOPERIMETRIC INEQUALITIES FOR A NONLINEAR EIGENVALUE PROBLEM

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ABSTRACT. An estimate for the spectrum of the two-dimensional eigenvalue problem $\Delta u + \lambda e^u = 0$ in D $(\lambda > 0)$, $u = 0$ on ∂D is derived, and upper and lower pointwise bounds for the solutions are constructed.

1. Let D be a simply connected bounded domain in the plane with a piecewise analytic boundary ∂D. Consider the nonlinear Dirichlet problem

$$\begin{aligned} \Delta u(x) + \lambda e^{u(x)} &= 0 \quad \text{in } D, \\ u(x) &= 0 \quad \text{on } \partial D, \end{aligned} \tag{1}$$

where λ is a *positive* real number and x stands for the generic point (x_1, x_2). This problem arises in the theory of self-ignition of a chemically active gas [4 and the literature cited there] and has been studied by many authors [2], [3], [5].

It was shown in [3] and [5] that there exists a number $\lambda^* > 0$ such that the problem has at least one solution for each $\lambda \leqslant \lambda^*$, but does not have solutions for $\lambda > \lambda^*$. Bounds for λ^* are found in [2]. In particular it was proved that $\lambda^* \geqslant 2\pi/A$ where A denotes the area of D. Equality is attained if and only if D is a circle. In this paper we prove that $\lambda^* \leqslant 2/R_0^2$, R_0 being the maximal conformal radius of D. We also give estimates for the solutions by means of the conformal radius. Our proofs are based on the introduction of a special system of coordinates defined by the level lines; see [6].

2. Let $g(x, \xi)$ be the Green's function for the Laplace operator, vanishing on ∂D. It is well known that for fixed $x \in D$

$$g(x, \xi) = (2\pi)^{-1} \log(R_x / |x - \xi|) + H(x, \xi) \tag{2}$$

where R_x is the *conformal radius of x with respect to D*,

$$|x - \xi| = \left(\sum_{i=1}^{2} (x_i - \xi_i)^2 \right)^{1/2}$$

and $H(x, \xi)$ is a harmonic function of ξ with $\lim_{\xi \to x} H(x, \xi) = 0$. With the help of this Green's function Problem (1) can be written as an integral equation:

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$$u(x) = \lambda \int_D g(x,\xi) e^{u(\xi)}\,d\xi, \quad \text{where } d\xi = d\xi_1\,d\xi_2. \tag{3}$$

We now keep x fixed and denote by $D(t)$ the domain $\{\xi \in D; g(x,\xi) > t\}$. It is homeomorphic to a circle. Let us assume that Problem (1) has a solution $u \in C^2(D) \cap C^0(\overline{D})$. Since e^t is real analytic, $u(x)$ is also real analytic. Define

$$a(t) = \int_{D(t)} e^{u(\xi)}\,d\xi.$$

Let $\Gamma(t) = \{\xi \in D; g(x,\xi) = t\}$. It is a simple closed curve and analytic for all $t > 0$. Denote by $\delta n > 0$ the piece of normal between $\Gamma(t)$ and $\Gamma(t + dt)$. If s is the arclength of $\Gamma(t)$, then [6, especially p. 213]

$$da(t) \equiv a(t + dt) - a(t) = -\oint_{\Gamma(t)} e^{u(\xi)}\,\delta n\,ds_\xi + o(dt).$$

Because of the strong maximum principle, $|\operatorname{grad} g(x,\xi)|$ cannot vanish on $\Gamma(t)$, hence

$$\frac{da}{dt} = -\oint_{\Gamma(t)} e^{u(\xi)} |\operatorname{grad} g(x,\xi)|^{-1}\,ds_\xi.$$

(3) can be written in the following form:

$$u(x) = -\lambda \int_0^\infty t \cdot \frac{da}{dt} \cdot dt. \tag{4}$$

Integration by parts yields

$$u(x) = \lambda \int_0^\infty a(t)\,dt. \tag{5}$$

By the Schwarz inequality we have

$$\oint_{\Gamma(t)} e^{u(\xi)}\,|\operatorname{grad} g|^{-1}\,d\xi \cdot \oint_{\Gamma(t)} |\operatorname{grad} g|\,ds_\xi \geqslant \left\{ \oint_{\Gamma(t)} e^{u(\xi)/2}\,ds_\xi \right\}^2$$

and therefore

$$-\frac{da}{dt} \geqslant \left\{ \oint_{\Gamma(t)} e^{u(\xi)/2}\,ds_\xi \right\}^2. \tag{6}$$

Consider the abstract surface given by the domain $D \subset \mathbf{R}^2$ and the Riemann metric $d\sigma^2 = e^{u(\xi)} ds^2$. Its Gaussian curvature is $K = -\Delta u/(2e^u) = \lambda/2$.

Because of the isoperimetric inequality for manifolds of constant Gaussian curvature [1, p. 514],

$$\left\{ \oint_{\Gamma(t)} e^{u(\xi)/2}\,ds \right\}^2 \geqslant 4\pi a(t) - \frac{\lambda}{2} a^2(t).$$

This inequality together with (6) implies that

$$-da/dt \geqslant 4\pi a(t) - \lambda a^2(t)/2.$$

Thus, $m(t) = e^{-4\pi t}(1/a(t) - \lambda/8\pi)$ is a nondecreasing function of t. From (2) we conclude that

$$\lim_{t\to\infty} m(t) = 1/(\pi R_x^2 e^{u(x)}).$$

Hence

$$m(t) \leqslant 1/(\pi R_x^2 e^{u(x)}) \quad \text{for all } t > 0,$$

$$a(t) \geqslant \frac{1}{e^{4\pi t}/(\pi R_x^2 e^{u(x)}) + \lambda/8\pi}.$$

If we insert this estimate into (5) and integrate, we obtain

$$e^{u(x)/2} \geqslant 1 + \lambda R_x^2 e^{u(x)}/8. \tag{7}$$

Let us put for short $\beta = \lambda R_x^2/8$. Then (7) yields

$$\left[e^{u(x)/2} - 1/(2\beta)\right]^2 + 1/\beta - 1/(4\beta^2) \leqslant 0. \tag{8}$$

Hence, the expression $1/\beta - 1/(4\beta^2)$ must be nonpositive, and we therefore have

$$\lambda R_x^2 \leqslant 2. \tag{9}$$

From this inequality we conclude that

$$\lambda^* \leqslant 2/R_0^2. \tag{10}$$

Consider now a circle of radius R. The radially symmetric solutions of (1) are in this case [4]

$$u_i(r) = \log \frac{b_i}{(1 + \lambda(b_i/8)r^2)^2}$$

where $r = |x|$ and

$$b_i = \frac{32}{\lambda^2 R^4}\left(1 - \frac{\lambda R^2}{4} + (-1)^i\left(1 - \frac{\lambda R^2}{2}\right)^{1/2}\right), \quad i = 1, 2.$$

the function $u_1(r)$ corresponds to the minimal solution [5], [2]. By [5, Theorem 3.2] it follows that Problem (1) has a solution if and only if a minimal solution exists. Thus, $\lambda^* = 2/R_0^2$.

We therefore have proved

THEOREM 1. *Let D be a simply connected domain in $\mathbf{R}^2$, and let R_0 be its maximal conformal radius. Then $\lambda^* \leqslant 2/R_0^2$. Equality holds for the circle.*

The next result is an immediate consequence of (8).

THEOREM 2. *Under the assumptions of Theorem* 1 *we have*

$$1 - \sqrt{1 - 4\beta} \leqslant 2e^{-u(x)/2} \leqslant 1 + \sqrt{1 - 4\beta} \tag{11}$$

where $\beta = \lambda R_x^2$.

Equality holds at the right-hand side if D is a circle, x is taken at the center and $u(x)$ is the minimal solution $u_1(r)$. Equality holds at the left-hand side if D is a circle, x is taken at the center and $u(x)$ corresponds to $u_2(r)$.

REMARK. Since $R_x \neq 0$ for $x \in D$, $x \notin \partial D$, (11) leads to the conjecture that for fixed λ all solutions of Problem (1) are uniformly bounded.

Let $d(x)$ be the distance from the point $x \in D$ to the boundary ∂D. By the monotony of R_x with respect to the domain it follows that $R_x \geqslant d(x)$. This inequality together with (11) leads to the

COROLLARY. *Under the assumptions of Theorem* 1 *we have*

$$1 - \sqrt{1 - \lambda d^2(x)/2} \leqslant 2e^{-u(x)/2} \leqslant 1 + \sqrt{1 - \lambda d^2(x)/2}.$$

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