

ON FUNCTIONS WITH SUMMABLE APPROXIMATE PEANO DERIVATIVE

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ABSTRACT. Let n be a positive integer and F a function defined on a closed interval I . For x in I , let the n th approximate Peano derivative of F at x , if it exists, be denoted as $F_{(n)}(x)$. For $n = 1$, the existence of $F_{(n-1)}(x)$ will simply mean that the function $F_{(0)} (= F)$ is approximately continuous at x . Then the following theorem is proved, noting that the phrase "for nearly all x in I " means "for all x in I except perhaps for those points in a countable subset of I ", **THEOREM A_n .** *Let $F_{(n-1)}(x)$ exist finitely for all x in I . If $F_{(n)}(x)$ exists finitely for nearly all x in I and is summable on I , then $F_{(n-1)}$ is absolutely continuous in I .*

1. For $n = 1$, Theorem A_n is clearly a generalization of the well-known result (cf. Goffman [1], where a short transparent proof was given): A function whose ordinary derivative exists everywhere and is summable is absolutely continuous. The proof to be given is somehow rather involved. In fact, we need the following two lemmas, which seem to be interesting by themselves.

LEMMA 1. *Let F be approximately continuous in I and*

$$\limsup \sup |F(x+h) - F(x)|/|h| < +\infty$$

for nearly all x in I . Then F is $[ACG]$ on I .

That F is $[ACG]$ on I means that I is the union of countably many *closed* sets on each of which the function F is AC (i.e., absolutely continuous in the wide sense, see Saks [6]). Note that Saks' definition of ACG does not require the sets involved to be *closed*. However, his definition requires the function to be continuous. This ensures that a function ACG in Saks' sense is $[ACG]$ since if a continuous function is AC on a set, it is AC on the closure of this set.

LEMMA 2. *Let H have the Darboux property and be $[ACG]$ on I . If $H_{(1)}(x) = 0$ for almost all x in I , then H is constant on I .*

Lemma 1 seems to be known for a long time (see Ridder [5]). We will give it a brief proof. Lemma 2 can be obtained from a theorem in [4], which gives an interesting characterization for monotone functions. However, we choose to give Lemma 2 a direct proof here.

We are unable to obtain a result similar to Lemma 1 for the proof of

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Theorem A_n for $n \geq 2$. Instead, for the proof to be given for $n \geq 2$, we will base on the following two results. The first one, Theorem 1, is known (see Saks [6, p. 191]) and, in fact, its application in the study of AC property for everywhere differentiable functions is also well known (see, e.g. Rudin [7]). The second one, Theorem 2, is recently due to the author [3].

THEOREM 1. *Let f be summable on I . Then for each $\varepsilon > 0$ there exists a lower semicontinuous summable function g such that $g(x) > -\infty$ and $g(x) \geq f(x)$ for all x in I and*

$$\int_c^d g(x) dx < \varepsilon + \int_c^d f(x) dx$$

for all c, d in I with $c < d$.

THEOREM 2. *Let $n \geq 2$ and suppose that $H_{(n-1)}(x)$ exists finitely for all x in I . If $uH_{(n)}(x) \geq 0$ for almost all x in I and $uH_{(n)}(x) \geq -\infty$ for nearly all x in I , or more generally, if $u_0 H_{(n)}(x) \geq 0$ for almost all x and $u_0 H_{(n)}(x) > -\infty$ for nearly all x in I , then $H_{(n-1)}$ is monotone increasing and continuous in I .*

Here, as in [3] $uH_{(n)}(x)$ is the n th upper approximate Peano derivate of H at x , while $u_0 H_{(n)}(x)$ denotes the extended real number obtained by taking in the definition of $uH_{(n)}(x)$ the ordinary lim sup instead of the approximate lim sup. Later on in the proof of Theorem A_n , we will also use $lH_{(n)}(x)$ and $l_0 H_{(n)}(x)$ to denote the corresponding lower values.

2. Proof of Lemma 1. For each positive integer n and each x in I , let $B(n, x) = \{y: |F(y) - F(x)| < n|y - x|\}$, and let E_n be the set of all x such that $|B(n, x) \cap J| > \frac{1}{2}|J|$ for all intervals $J \subset I$ with x in J and $|J| < 1/n$. Then as F is approximately continuous in I , one shows that for any x_1, x_2 in the closure of E_n with $|x_2 - x_1| < 1/n$,

$$|F(x_2) - F(x_1)| \leq n|x_2 - x_1|.$$

Hence, F is AC on the closure of each E_n .

Now, let E be the set of all points x such that

$$\limsup |F(x+h) - F(x)|/|h| < +\infty.$$

Then the set $S = I \sim E$ is at most countable, say $S = \{y_1, y_2, y_3, \dots\}$, and denote $S_j = \{y_j\}$ for $j = 1, 2, 3, \dots$, F is AC on each S_j since it is a singleton. On the other hand, each x in E is a point in E_n for some n . Hence, letting C_n be the closure of E_n for all n , one has $I = [\cup C_n] \cup [\cup S_j]$, a union of countably many closed sets on each of which F is AC.

3. Proof of Lemma 2. Let G be the set of all points x in I such that H is constant on $J_x \cap I$ for some open interval J_x containing x . Then G is open in I . Suppose $G \neq I$. We will reach a contradiction. To see this, consider the nonempty closed set $E = I \sim G$. Since H is [ACG] on I , it follows from the Baire category theorem that there exists $(c, d) \subset I$ such that $(c, d) \cap E \neq \emptyset$ and H is AC on $[c, d] \cap E \equiv P$. Note that in each closed interval contained in $[c, d] \sim P$, the function H is constant. Hence in the closure of each interval

contiguous to P (with respect to $[c, d]$), H is constant since H has the Darboux property. Then it follows easily that H , being AC on the closed set P , is AC on the whole interval $[c, d]$. Now, using this fact as well as that $H_{(1)}(x) = 0$ for almost all x in $[c, d]$, we show that H is constant on $[c, d]$. It clearly suffices to show that $H(d) = H(c)$. To this end, let $\varepsilon > 0$ be given but arbitrary, and let A be the set of all x in $[c, d]$ such that there exists a sequence of points x_i with $x_i \neq x$ and $x_i \rightarrow x$ such that $|H(x_i) - H(x)| < \varepsilon |x_i - x|$. Then $|A| = d - c$ since $H_{(1)}(x) = 0$ for almost all x in $[c, d]$. Furthermore, the family of all intervals with x and x_i as the end points covers A in the Vitali sense. Hence for each $\delta > 0$, there exists a finite set $\{[y_k, y'_k]\}$ of mutually exclusive intervals from the family such that $\sum (y'_k - y_k) \geq (d - c) - \delta$. Then $\sum (t'_j - t_j) < \delta$, where $\{(t_j, t'_j)\}$ is the set of all subintervals of $[c, d]$ complementary to the set $\{[y_k, y'_k]\}$. As H is AC on $[c, d]$, choosing δ small enough, one has

$$\sum |H(t'_j) - H(t_j)| < \varepsilon.$$

Hence

$$\begin{aligned} |H(d) - H(c)| &\leq \sum |H(y'_k) - H(y_k)| + \sum |H(t'_j) - H(t_j)| \\ &\leq \varepsilon \sum (y'_k - y_k) + \varepsilon \leq \varepsilon [d - c + 1]. \end{aligned}$$

As ε is arbitrary, we conclude that $H(d) = H(c)$. Thus, we have proved that H is constant on $[c, d]$, contradicting the fact that $(c, d) \cap E \neq \emptyset$. Hence we must have $G = I$. Then, using the Heine-Borel theorem, one sees easily that H is constant on I , completing the proof.

4. Proof of Theorem A_n . Let $K(x) = \int_a^x F_{(n)}(t) dt$ for all x in I , where a is the left end point of I . Then it is well known that K is AC on I and $K'(x) = F_{(n)}(x)$ for almost all x in I .

(i) For the case $n = 1$, consider the function $H = K - F$. Clearly, H is approximately continuous in I and hence H has the Darboux property in I . Furthermore, noting that F is [ACG] on I by Lemma 1, one sees that H , as a difference of an AC and an [ACG] function, is [ACG] on I . It then follows from Lemma 2 that H is constant on I since $H_{(1)}(x) = K'(x) - F_{(1)}(x) = 0$ for almost all x in I . Hence $F (= K - H)$ is AC on I .

(ii) For the cases $n \geq 2$, let $f = F_{(n)}$, and for each $\varepsilon > 0$ let g be as in Theorem 1. Then consider the function $H = G - F$, where G is a function such that its $(n - 1)$ th ordinary derivative $G^{(n-1)}(x) = M(x) \equiv \int_a^x g(t) dt$. Then $H_{(n-1)} = G^{(n-1)} - F_{(n-1)} = M - F_{(n-1)}$ on I . We will show further that $u_0 H_{(n)}(x) \geq 0$ for nearly all x in I (see the remark following Theorem 2 for the notations involved). To this end, note first that $l_0 M_{(1)}(x) \geq g(x)$ for all x in I since g is lower semicontinuous in I . Furthermore, by a simple application of the following easy inequality (see Hobson [2])

$$\liminf \frac{A(h)}{B(h)} \geq \liminf \frac{A'(h)}{B'(h)},$$

one has $l_0 G_{(n)}(x) \geq l_0 M_{(1)}(x)$ for all x in I . Hence

$$\begin{aligned}
u_0 H_{(n)}(x) &\geq uH_{(n)}(x) \geq lG_{(n)(x)-uF_{(n)}}(x) \\
&\geq l_0 G_{(n)}(x) - uF_{(n)}(x) \geq l_0 M_{(1)}(x) - uF_{(n)}(x) \\
&\geq g(x) - uF_{(n)}(x)
\end{aligned}$$

for all x in I . But $uF_{(n)}(x) = F_{(n)} \neq \pm\infty$ for nearly all x in I , so that one has that $u_0 H_{(n)}(x) \geq g(x) - F_{(n)}(x) \geq 0$ for nearly all x in I . Therefore, applying Theorem 2, one has that $H_{(n-1)}$ is monotone increasing in I . Hence

$$F_{(n-1)}(d) - F_{(n-1)}(c) \leq M(d) - M(c) < \varepsilon + \int_c^d f(t) dt$$

for all c, d in I with $c < d$. But since ε is arbitrary, one concludes that

$$F_{(n-1)}(d) - F_{(n-1)}(c) \leq \int_c^d f(t) dt = K(d) - K(c).$$

Replacing, in the above argument, F by $-F$, one obtains the inverse of the above inequality. Hence one concludes that the above inequality is, in fact, an equality. Therefore, $F_{(n-1)}$ is AC on I since K is.

5. Further remarks and acknowledgement. It is clear that Lemma 1 and Theorem 2 are more general than what is required for the proof of Theorem A_n . If the full force of both results are used, a slight modification of the proof of Theorem A_n yields the following improvement.

THEOREM B_n . *Let $F_{(n-1)}(x)$ exist finitely for all x in I , and $F_{(n)}(x)$ exist finitely for almost all x in I and be summable on I . If $uF_{(n)}(x) < +\infty$ and $lF_{(n)}(x) > -\infty$ for nearly all x in I , then $F_{(n-1)}$ is absolutely continuous in I .*

We remark that Bagby and Ziemer in a recent paper [8] have mentioned that C. J. Neugebauer has proved the following result: A function whose approximate derivative exists everywhere and is summable is absolutely continuous. This is slightly less general than Theorems A_0 or B_0 here. On the other hand, in their paper [8] concerning functions of several real variables, Bagby and Ziemer have also established, among others, a result containing Neugebauer's result. The comparison of the results here and some in [8] for the special case of functions of a single real variable seems worthwhile to make a further detailed study in another paper.

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