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SUBNORMAL SHIFTS WITH OPERATOR-VALUED WEIGHTS[1]

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ABSTRACT. A criterion for a shift with operator-valued weights to be subnormal is given. From this a different proof of a theorem of Berger and Sarason is deduced.

The purpose of this note is to give an extension of a well-known theorem due to Berger and Sarason [2, p. 895]. A similar result was obtained by Gellar and Wallen [3]. Our method gives a new proof of their result.

THEOREM 1. *Let $\mathcal{H}_0$ be a Hilbert space, $\{A_i\}_{i=0}^{\infty}$ be a sequence of nonnegative invertible operators on $\mathcal{H}_0$ satisfying* $\sup\|A_i\| = 1$. *Then the operator T on $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_0 \oplus \cdots$ defined by $T\{x_0, x_1, \ldots\} = \{0, A_0 x_0, A_1 x_1, \ldots\}$ is subnormal if and only if there exists a positive operator valued measure Φ on $[0, 1]$ taking values in $\mathcal{L}(\mathcal{H}_0)$ such that for $n \geqslant 0$, $B_n^* B_n = \int_0^1 t^n d\Phi(t)$ where $B_0 = 1$ and $B_n = A_{n-1} \cdots A_0$ for $n \geqslant 1$. Moreover whenever T is subnormal, there exists a Hilbert space $\mathcal{K}_0$ containing $\mathcal{H}_0$ and a positive operator R on $\mathcal{K}_0$ such that the minimal normal extension S of T is given by $S\{y_n\}_{-\infty}^{\infty} = \{z_n\}_{-\infty}^{\infty}$ where $y_0 \in \mathcal{H}_0$, $y_n \in \overline{\text{Range}\, R}$ for $n \neq 0$ and $z_n = Ry_{n-1}$ for all n.*

PROOF. If T is subnormal and S is a normal extension of T acting on $\mathcal{K}$, such that $\|S\| = \|T\| = 1$, let $E(t)$ be a spectral measure defined on $[0, 1]$ such that $S^* S = \int_0^1 t\, dE(t)$ and let $\Phi(t) = P_0 E(t) P_0$ where P_0 is the projection of $\mathcal{K}$ on $\mathcal{H}_0$. Then for x_0, y_0 in $\mathcal{H}_0$,

$$\begin{aligned}(B_n^* B_n x_0, y_0) &= (T^n\{x_0, 0, \ldots\}, T^n\{y_0, 0, \ldots\}) \\ &= (S^n S^n\{x_0, 0, \ldots\}, \{y_0, 0, \ldots\}) = \int t^n d(\Phi(t) x_0, y_0).\end{aligned}$$

Hence $B_n^* B_n = \int_0^1 t^n d\Phi(t)$.

Conversely if $\Phi(t)$ is a positive operator valued measure on $[0, 1]$, by Naimark's theorem [1, p. 74] there exists a Hilbert space $\mathcal{K}_0$ containing $\mathcal{H}_0$ and a spectral measure $E(t)$ on $[0, 1]$ such that if P_0 is the projection of $\mathcal{K}_0$ on $\mathcal{H}_0$, $\Phi(t) = P_0 E(t) P_0$. Let $R = \int_0^1 t^{1/2} dE(t)$. For $n \geqslant 0$, define $U_n\colon \mathcal{H}_0 \to \mathcal{K}_0$ by $U_n(x_0) = R^n B_n^{-1} x_0$. Since

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$$\|B_n x_0\|^2 = (B_n^* B_n x_0, x_0) = \int_0^1 t^n d(\Phi(t)x_0, x_0)$$
$$= \int_0^1 t^n d(E(t)x_0, x_0) = \|R^n x_0\|^2,$$

it follows that U_n is an isometry. Let $\mathcal{K} = \{\{y_n\}_{-\infty}^{\infty}, y_n \in K_0$ for $n \neq 0, y_0 \in \overline{\text{Range } R}\}$ and define S on $\mathcal{K}$ by $S\{y_n\}_{-\infty}^{\infty} = \{Ry_{n-1}\}_{-\infty}^{\infty}$. Then S is a bounded normal operator. We now seek an embedding U: $\mathcal{H} \to \mathcal{K}$ such that $SU = UT$. Let U be defined by $U\{x_0, x_1, x_2, \ldots\} = \{z_n\}_{-\infty}^{\infty}$ where $z_n = 0$ for $n < 0$, $z_n = U_n x_n$ for $n \geqslant 0$. Clearly U is an isometry. Since $U_n B_n = R^n$ and $A_n = B_{n+1} B_n^{-1}$, it follows that $SU = UT$. It is easy to see that if we choose E to be the minimal dilation of Φ, S becomes the minimal normal extension of T.

REMARK. When $\mathcal{H}_0$ is the space of complex numbers, S is easily seen to be equivalent to the normal operator defined by Berger (see [2, pp. 895–896]).

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