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COUNTABLE PARACOMPACTNESS, NORMALITY, AND MOORE SPACES

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ABSTRACT. In this paper we show that MA + $\neg$ CH implies that there exists a countably paracompact Moore space which is not normal. Further, if there is a model of set theory in which every countably paracompact Moore space is normal, then the normal Moore space conjecture is true in that model. Other examples are given, including a nonnormal space constructed with $\diamondsuit$ which is countably compact, T_3, first countable, locally compact, perfect, and hereditarily separable.

Introduction. There are many examples of countably paracompact spaces which are not normal. For example $(\omega_1 + 1) \times \omega_1$ is even countably compact but not normal. However, it seems much more difficult to construct an example of a Moore space or, more generally, a perfect space which is countably paracompact but not normal. In this paper we give examples of such spaces, thus answering, or partially answering, a number of open questions of [2], [7]–[10], and [12].

In §1, we describe a machine that takes an arbitrary normal, noncollectionwise normal space X, and from it produces a regular, nonnormal space, X^*. Further properties of X^* can be deduced from properties of X. In particular, if X is countably paracompact, perfect, Moore, first countable, or collectionwise Hausdorff, then X^* also has the corresponding property. In §2, we present corollaries of the methods of §1 that relate to Moore spaces. Non-Moore examples are given in §3, including an example which proves false the converse of Dowker's 1951 theorem stating that in perfect spaces, normality implies countable paracompactness.

By a *space*, we will always mean a Hausdorff space. A space is called *perfect* if each closed set is G_δ. MA + $\neg$ CH stands for *Martin's axiom plus the negation of the continuum hypothesis*, while we abbreviate *collectionwise normal* by CWN and *collectionwise Hausdorff* by CWH. Terms not defined above can be found in [9].

1. **The machine.** We now describe a general machine that takes a normal, noncollectionwise normal space, X, and associates with it a regular, nonnormal space which we will call X^*. The machine constructs X^* in such a way that the following theorem is true.

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THEOREM. *If X is normal but not CWN, then X^* is a regular, nonnormal space and*

(1) *X countably paracompact $\to X^*$ countably paracompact.*

(2) *X Moore $\to X^*$ Moore.*

(3) *X first countable $\to X^*$ first countable.*

(4) *X CWH $\to X^*$ CWH.*

(5) *X perfect $\to X^*$ perfect.*

Although we will only be concerned with the above properties in this paper, it is easy to prove that the machine preserves many other properties (including metacompactness and orthocompactness) and does not increase the value of many of the cardinal functions (such as character and pseudocharacter).

Suppose X is a normal, not CWN space and $\{H^\alpha\}_{\alpha<\xi}$ is a discrete collection of closed sets which witnesses the non-CWN of X. Let $D = X - H$, where $H = \bigcup\{H^\alpha: \alpha < \xi\}$.

Let

$$X^* = (X \times \{0, 1\}) \cup (D \times \{(\alpha, \beta): \alpha, \beta < \xi \text{ and } \alpha \neq \beta\}).$$

If $A \subset X$ and $\delta \in \{0, 1\} \cup \{(\alpha, \beta): \alpha, \beta < \xi \text{ and } \alpha \neq \beta\}$, we will use A_δ to denote $(A \times \{\delta\}) \cap X^*$. Similarly, if $x \in X$ then x_δ denotes $\{x\} \times \{\delta\}$.

Each point of $X^* - (H_0 \cup H_1)$ is declared to be a basic open set in X^*. For each U open in X and each $\alpha < \xi$ such that U is contained in $H^\alpha \cup D$, we define each of the following to be basic open subsets of X^*:

$$\bigcup \{U_{(\alpha,\beta)}: \alpha \neq \beta < \xi\} \cup U_0 \quad \text{and} \quad \bigcup \{U_{(\beta,\alpha)}: \alpha \neq \beta < \xi\} \cup U_1.$$

The construction of X^* is completed by giving X^* the topology generated by the base described above.

PROOF OF THEOREM. The regularity of X^* follows from the regularity of X and the fact that if U is open in X and $U \subset H^\alpha \cup D$, then the two open sets derived above from U and α are disjoint. (Of course, this also implies that X^* is Hausdorff.)

Next we show that X^* is not normal. Suppose U and V are open sets in X^* that contain X_0 and X_1 respectively. For $A \subset X^*$, define $\pi(A) = \{x \in X: \{x\} \times \{\delta\} \in A$ for some $\delta\}$.

For $\alpha < \xi$, choose *basic* open sets $U^\alpha \subset U$ and $V^\alpha \subset V$ containing H_0^α and H_1^α respectively. Without loss of generality, $\pi(U^\alpha) = \pi(V^\alpha)$ for all $\alpha < \xi$. By the choice of the H^α's, there exist $\alpha, \beta < \xi$ such that $\alpha \neq \beta$ and $U^\alpha \cap U^\beta \neq \varnothing$. But now this implies $U^\alpha \cap V^\beta \neq \varnothing$ so we can conclude that X^* is not normal.

To verify (1), suppose that X is countably paracompact and $\{F^n\}_{n\in\omega}$ is a decreasing sequence of closed sets in X^* whose intersection is empty. We prove that X^* is countably paracompact by finding a family $\{V^n\}_{n\in\omega}$ of open sets such that $F^n \subset V^n$ and $\bigcap\{\mathrm{cl}(V^n): n \in \omega\} = \varnothing$. Without loss of generality, $F^0 \subset X_1 \cup X_2$. Since X is countably paracompact and $\bigcap\{\pi(F^n): n \in \omega\} = \varnothing$, we can find open sets $U^n \supset \pi(F^n)$ for all $n \in \omega$ such that $\bigcap\{\mathrm{cl}(U^n): n \in \omega\} = \varnothing$. Now $\pi^{-1}(U^n) \supset F^n$ and $\{\pi^{-1}(U^n)\}_{n\in\omega}$ is a collection of open sets in X^* whose closures have empty intersection, so X^* is countably paracompact.

The verification of (2)–(5) is similar and is left to the reader.

2. **Moore spaces.** R. H. Bing's theorem [1] that a Moore space is metrizable if and only if it is collectionwise normal immediately gives us the following corollary of the Theorem of §1.

COROLLARY 1. *If there exists a normal nonmetrizable Moore space, then there exists a countably paracompact Moore space which is not normal.*

Since MA + $\neg$CH implies that there exists a nonmetrizable normal Moore space (see [9]) we have

COROLLARY 2. MA + $\neg$CH *implies that there exists a countably paracompact Moore space which is not normal.*

P. Zenor has shown [12] that Corollary 2 is equivalent to each of the following two facts.

COROLLARY 3. MA + $\neg$CH *implies that there exists a countably paracompact Moore space which is not hereditarily countably paracompact.*

COROLLARY 4. MA + $\neg$CH *implies that there exists a closed continuous function from a countably paracompact Moore space onto a space which is not countably paracompact.*

Corollary 4 is especially interesting since it is known that closed maps preserve normality, paracompactness, and in the presence of normality, λ-paracompactness. It is also known that the closed image of a countably paracompact space is countably paracompact if it is first countable.

W. G. Fleissner has shown [4] that it is consistent with ZFC that there exist a normal, CWH, nonmetrizable Moore space. Since every normal Moore space is countably paracompact, this gives us

COROLLARY 5. *CON. (There exists a countably paracompact, CWH, nonnormal Moore space.)*

Fleissner [4] (also Alster and Pol) has already given a consistent example of a CWH, nonnormal Moore space. Corollary 5 strengthens this result with the addition of countable paracompactness and gives us an alternate method of construction. Note that an absolute example of a nonnormal Moore space that is CWH (but not countably paracompact) will appear in [11].

We end this discussion of Moore spaces with the following questions:

(1) Assuming $V = L$, is every countably paracompact Moore space normal?

(2) Does the existence of a countably paracompact nonnormal Moore space imply the existence of a normal nonmetrizable Moore space?

A positive answer to the first question would of course imply that the normal Moore space conjecture is true under $V = L$. Thus the normal Moore space conjecture would be independent of the usual axioms for set theory. Fleissner [4] has proven that CH implies that every *separable* countably paracompact Moore space is normal.

3. **Other examples.** Example H of [1] is a perfect, countably paracompact, normal, not CWN space that is constructed without the aid of set theoretic assumptions beyond the axiom of choice. Hence without the extra assumptions of Corollary 2, we still have

COROLLARY 6. *There is a perfect, T_3 space which is countably paracompact but not normal.*

As in §2, Zenor's work [12] and Corollary 6 give us a real example of a countably paracompact T_3 space which is not hereditarily countably paracompact and show us that a closed map from a regular, perfect space need not preserve countable paracompactness.

AN EXAMPLE USING $\diamondsuit$. The last example of this paper, X, is another countably paracompact space which is not normal. Despite its nonnormality, X does have a number of amazingly strong properties. In fact, X is countably compact, perfect, T_3, first countable, locally compact, locally countable, zero dimensional, and hereditarily separable.

The following construction of X depends heavily on Jensen's $\diamondsuit$[3] and the technique used by A. J. Ostaszewski [6] to construct a perfectly normal, countably compact, hereditarily separable T_3 space which is not compact. $\diamondsuit$ is a combinatorial statement that is independent of the usual axioms for set theory since it implies CH and is implied by $V = L$. We will use $\diamondsuit$ in the equivalent form of $\clubsuit$ + CH.

Define $X = \omega_1 \times \omega$, $X_\alpha = \alpha \times \omega$ for all $\alpha < \omega_1$, $C_n = \omega_1 \times \{n\}$ for all $n \in \omega$ and $\Lambda = \{\lambda < \omega_1: \lambda \text{ is a limit ordinal}\}$. We will construct the topology on X so that each C_n is a closed set that is homeomorphic to Ostaszewski's example. For each $n > 1$, C_n will be open in X. However, an open set containing C_0 or C_1 will be forced to contain a large part of $\bigcup\{C_n: 1 < n \in \omega\}$ so that C_0 cannot be separated from C_1.

To make X countably compact, we will give each countably infinite subset of X a limit point. With this goal in mind, use the continuum hypothesis to fix an enumeration $\langle S_\lambda: \lambda \in \Lambda\rangle$ of all countably infinite subsets of X such that for each $\lambda \in \Lambda$, either

(a) $|S_\lambda \cap C_n| \leqslant 1$ for all $n \in \omega$, or

(b) $S_\lambda \subset C_n$ for some $n \in \omega$.

Without loss of generality, we assume $\sup S_\lambda < \lambda$. To keep track of the S_λ, define the function $j: \Lambda \to \omega$ by $j(\lambda) = 0$ if S_λ satisfies (a) above and $j(\lambda) = n$ if $S_\lambda \subset C_n$.

$\clubsuit$ will be used to help make X perfect, hereditarily separable and nonnormal. $\clubsuit$ states that there exists a family $\{M_\lambda: M_\lambda \subset \lambda \in \Lambda\}$ such that each M_λ is a simple increasing sequence in ω_1, cofinal in λ and for each uncountable $M \subset \omega_1$, there is a $\lambda \in \Lambda$ with $M_\lambda \subset M$. Without loss of generality we assume $\sup S_\lambda < \inf M_\lambda$ for all $\lambda \in \Lambda$. In connection with the nonnormality of X it will be convenient to fix a function $p: \Lambda \to 2$ such that both $p^{-1}(\{0\})$ and $p^{-1}(\{1\})$ are uncountable.

We are now ready to build topologies T_λ on X_λ inductively for all $\lambda \in \Lambda$ so that whenever $\xi \in \Lambda \cap \lambda$, $\alpha \in \lambda$ and $n \in \omega$, we have

(1) $T_\xi = T_\lambda \cap \mathcal{P}(X_\xi)$.

(2) Each T_λ is first countable, T_2 and locally compact.

(3) X_α is an open subset of (X_λ, T_λ).

(4) $C_n \cap X_\lambda$ is closed in (X_λ, T_λ) for all $n \in \omega$, and open in (X_λ, T_λ) for $n > 1$.

Let T_ω be the discrete topology on ω. For $\omega < \eta \in \Lambda$, we assume that we have defined T_λ for all $\lambda \in \eta \cap \Lambda$ so that (1)–(4) hold and we show how to define T_η so that (1)–(4) hold for all $\lambda \in (\eta + 1) \cap \Lambda$.

If η is a limit of Λ, condition (1) forces us to take T_η to be the topology generated by $\bigcup\{T_\xi: \xi \in \eta \cap \Lambda\}$. It is easy to check that (1)–(4) are preserved by this definition.

Now suppose $\eta = \xi + \omega$ for some $\xi \in \Lambda$. For each $n \in \omega$, let $D_n = M_\xi \times \{n\}$ if $n \neq j(\xi)$ or S_ξ has a limit point in (X_ξ, T_ξ), and let $D_{j(\xi)} = (M_\xi \times \{j(\xi)\}) \cup S_\xi$ if S_ξ has no limit point in (X_ξ, T_ξ). Enumerate each D_n as $\langle d(k, n): k \in \omega\rangle$ and enumerate M_ξ as $\langle m(k, \xi): k \in \omega\rangle$ so that $m(k, \xi) < m(k + 1, \xi)$. Condition (2) implies that (X_ξ, T_ξ) is a zero-dimensional metric space, and each D_n is closed in (X_ξ, T_ξ) by condition (3). Thus by condition (4) for all $k, n \in \omega$ we can choose $U(k, n)$ and $V(k)$ so that

(a) $U(k, n)$ and $V(k)$ are compact open subsets of X_ξ that contain $d(k, n)$ and $(m(k, \xi), k)$ respectively,

(b) $\bigcup\{U(k, n): k \in \omega\}$ and $\bigcup\{V(k): k \in \omega\}$ are closed in X_ξ for all $n \in \omega$, and hence $\{U(k, n): k \in \omega\}$ and $\{V(k): k \in \omega\}$ are discrete in X_ξ,

(c) whenever $n = 0$ or 1 and $m \in \omega$, we have $V(k) \cup U(k, n)$ contained in $X - \bigcup\{C_n: 2 \leqslant n \leqslant m\}$ for all but finitely many $k \in \omega$, and

(d) whenever $1 < n \in \omega$ and $k \in \omega$, $U(k, n) \subset C_n$.

Partition ω into ω infinite pieces, P_i, $i \in \omega$. We are now ready to define neighborhoods for the points in $X_\eta - X_\xi$. For each $i, n \in \omega$ such that $(i, n) \neq (0, p(\xi))$ we define the mth basic neighborhood of the point $(\xi + i, n) \in X_\eta$ to be

$$B(\xi + i, n, m) = \bigcup \{U(k, n): k \in P_i, m < k\} \cup \{(\xi + i, n)\}.$$

The mth basic neighborhood of $(\xi, p(\xi))$ is defined to be

$$B(\xi, p(\xi), m) = \bigcup \{U(k, n): k \in P_0, m < k\} \cup \bigcup \{V(k): m < k\} \cup \{(\xi, p(\xi))\}.$$

Finally, define T_η to be the topology whose basis is $\bigcup\{B(\xi + i, n, m): i, n, m \in \omega\} \cup T_\xi$. Then T_η satisfies (1)–(4) and the successor step is completed.

If we now define $T = T_{\omega_1}$, then T is a locally compact, T_2, first countable topology on X. Also, since each countably infinite subset of X contains S_λ for some λ, and each S_λ was given a limit point, X is countably compact. Before we verify other properties of X, let us prove the

Claim. If D is a closed subset of X, then for each $n \in \omega$, either $D \cap C_n$ or $C_n - D$ is countable.

PROOF. If $D \cap C_n$ is uncountable, then $♣$ implies that $M_\lambda \times \{n\}$ is contained in $D \cap C_n$ for some $\lambda \in \Lambda$. But by construction $(\omega_1 - \lambda) \times \{n\}$ is contained in the closure of $M_\lambda \times \{n\}$ so $C_n - D$ must be countable.

An examination of the above proof quickly leads us to the fact that X is hereditarily separable. The claim, together with the fact that any union of C_n's is trivially a G_δ in X implies that X is perfect.

Finally, we prove that C_0 and C_1 cannot be separated by disjoint open sets, so that X is not normal. Suppose U_0 and U_1 are disjoint open sets containing C_0 and C_1 respectively. Since by construction U_0 must contain a tail of the sequence $\langle(m(n, \xi), n): n \in \omega\rangle$ for uncountably many ξ, it follows from the claim that there exists a $k \in \omega$ and $\alpha \in \omega_1$ such that $\{(\beta, i): \alpha < \beta$ and $k < i\} \subset U_0$. Since by the same reasoning U_1 must also contain such a "tail" of X, $U_0 \cap U_1 \neq \varnothing$ and X cannot be normal.

References

1. R. H. Bing, *Metrization of topological spaces*, Canad. J. Math. **3** (1951), 175–186. MR **13**, 264.

2. R. H. Bing and R. J. Bean (Editors), *Topology seminar* 1965, Ann. of Math. Studies, no. 60, Princeton Univ. Press, Princeton, N. J., 1966. MR **34** #1974; erratum, **35**, p. 1577.

3. K. J. Devlin and H. Johnsbiaten, *The Souslin problem*, Lecture Notes in Math., vol. 405, Springer-Verlag, New York, 1974.

4. W. G. Fleissner, *Separation properties in Moore spaces* (to appear).

5. R. W. Heath, *Screenability, pointwise paracompactness, and metrization of Moore spaces*, Canad. J. Math. **16** (1964), 763–770. MR **29** #4033.

6. A. J. Ostaszewski, *On countably compact, perfectly normal spaces*, J. London Math. Soc. (to appear).

7. G. M Reed, *Examples of non-normal perfect spaces in which countable paracompactness is undetermined*, talk at Memphis State Univ. Topology Conf., 1975.

8, ——— , *On normality and countable paracompactness* (to appear).

9. M. E. Rudin, *Lectures on set theoretic topology*, Conference Board Math. Sci. Regional Conf. Ser. in Math., no. 23, Amer. Math. Soc., Providence, R. I., 1975.

10. M. K. Singal and Shasi Prabha Arya, *A survey of countably paracompact spaces*, Math. Student **11** (1972), 337–372.

11. M. L. Wage, *A collectionwise Hausdorff non-normal Moore space*, Canad. J. Math. (to appear).

12. P. Zenor, *On countable paracompactness and normality*, Prace Mat. **13** (1969), 23–32. MR **40** #1975.

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